INFORMATION AND COMPUTATION **88**, 18–59 (1990)

# Easy Multiplications
# II. Extensions of Rational Semigroups*

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In our first part of this paper we defined and studied the family of *rational monoids*, which are monoids with an "easy" multiplication (by this we mean indeed that the multiplication can be realized by a finite automaton). In this second part of our work, we consider *rational semigroups* (instead of monoids) and investigate the properties of two operations on these semigroups: the *Redei extensions* and the *ideal extensions*. We give necessary and sufficient conditions for such extensions of rational semigroups to be rational semigroups. The theory of ideal extensions is also used to build a non-rational monoid in which Kleene's theorem holds, which answers a question asked in the first part. A last application of our work on Redei and ideal extensions is the construction of a family of monoids "too simple" to be syntactic monoids of context-free languages. 

In the first part of this paper (Sakarovitch, 1987), we have defined and studied a family of monoids which we called *rational monoids* and of which we said that the multiplication is *easy* (by this we mean indeed that the multiplication can be realized by a finite transducer). We established in particular an algebraic property of these rational monoids, expressed in terms of Green's relations, and we showed that Kleene's theorem holds in every rational monoid. In the last section of that first part, we gave the necessary and sufficient conditions for the direct product—respectively Rees quotient and free product—of two rational monoids to be rational. In this second part of our work on rational monoids, we investigate the properties of two other closure operations on rational monoids: the *Redei extensions* (which are generaliations of direct product) and the *ideal extensions* (which give the inverse operation of Rees quotient).

Our first motivation for dealing with such constructions is indeed the problem that leads us to the definition of rational monoids. In the paper where he initiated the study of syntactic monoids of context-free languages, Perrot (1977) gave several examples of monoids that are not syntactic

* This work has been partially supported by the Programme de Recherches Coordonnées "Mathématiques et Informatique" of the Ministère de la Recherche et de la Technologie.



0890-5401/90 $3.00

monoids of any context-free language. It was clear then that these examples were not all of the same kind. Intuitively, some were "too complicated," whereas some others—a family of them named "monoïdes filiformes," threadlike monoids in English—were "too simple" to be syntactic monoids of context-free languages. We get to the definition of rational monoids as we wanted to describe formally the intuition we had on what is a "simple" monoid. We had then to verify that threadlike monoids belong indeed to that family of "simple" monoids. By definition, they are (some restricted kinds of) ideal extensions of a finite monoid by a Redei extension of **N** by a finite semigroup. Thus, from the beginning, we had to study Redei and ideal extensions of rational monoids and we prove here that rational monoids are closed (under certain conditions that are fulfilled by threadlike monoids) under these operations. Hence threadlike monoids are rational and the loop is looped.

The second motivation for such a study is to go further in the elucidation of the structure of rational monoids. As a by-product of the work conducted here we get an answer to a problem that we stated in the first part: we give here an example of a monoid which is not rational and in which Kleene's theorem still holds.

The paper contains two mains sections: Section 2 on Redei extensions and Section 3 on ideal extensions, organized in the same way. We first recall the more or less classical algebraic theory of the considered extension. We then construct a description (with the meaning we gave to that object in Part I, cf. also Section 1 below) of the extension from the descriptions of the two monoids or semigroups that compose the extension. We finally characterize those extensions that are rational, i.e., extensions with rational descriptions.

In (Redei, 1952), Redei has presented a construction that generalizes to monoids Schreier's theory of extensions of groups. Direct and semidirect products are particular cases of that construction. Redei extensions of monoids are not necessarily monoids (cf. Example 2.3); that is the reason why we consider semigroups rather than monoids. Redei extensions of rational semigroups are not necessarily finitely generated and thus not rational. This property of finite generation happens to be the crucial one and we finally establish the following:

THEOREM 2.3. *A Redei extension of a rational semigroup by a finite semigroup is a rational semigroup if and only if it is finitely generated.*

The core of this result is the fact that the mechanism of a Redei extension can be realized by the output function of a finite transducer (Lemma 2.4).

An ideal extension is a construction inverse ot the Rees quotient: if $M$ and $N$ are two semigroups, a semigroup $U$ is an ideal extension of $N$ by $M$

if it contains $N$ as an ideal and if the Rees quotient $U//N$ is isomorphic to $M^0$ ($M^0$ is equal to $M$, if $M$ has a zero, and to $M$ with an adjoint zero, otherwise). The classical semigroup theory (cf. Petrich, 1973) is mainly concerned with the problem of determining all the ideal extensions of $N$ by $M$, when $M$ and $N$ are given semigroups, and of characterising those which are isomorphic. Our point of view here is somewhat different since we are interested to know whether this operation of ideal extension preserves rationality or not. As it is done in Petrich (1973), an ideal extension is defined by a pair of functions: one called representation and the other ramification. For our purpose, we transfer these functions to functions between free monoids and the extension is said to be regular if the transferred ramification is a rational function. We then establish:

THEOREM 3.1. *An ideal extension of a rational semigroup by a rational semigroup is a rational semigroup if and only if it is a regular ideal extension.*

This implies that an extension of a rational semigroup $N$ by a rational semigroup $M$ is always rational if $M$ is finite or without zero or if $N$ is a monoid or a free semigroup (Corollaries 3.4, 3.2, 3.3, and 3.6). In Part I we stated that: *any Rees quotient of a rational monoid by a rational ideal is a rational monoid* (Proposition I.6.1). Theorem 3.1 is not a total converse of Proposition I.6.1, since an ideal of a rational monoid (or semigroup) does not need to be a finitely generated semigroup and since a non-finitely generated semigroup is not a rational semigroup. The general case is thus *a priori* out of the scope of the techniques we have developed here. It may be treated in a broader framework where the multiplication of an infinitely generated semigroup can be realized by a finite transducer (Pelletier, 1989). However, Theorem 3.1 implies that a rational monoid $M$ is a regular ideal extension of any ideal $I$ by the Rees quotient $M//I$, as soon as $I$ is finitely generated as a semigroup of $M$, which makes the whole theory consistent. As we said before, rational manoids have the remarkable property that they are all monoids in which Kleene's theorem holds. Such monoids have been studied for their own sake (Reutenauer, 1985). Moreover, every example of monoid known so far in which Kleene's theorem holds is a rational monoid as we noted in Part I. The machinery we have set up to deal with ideal extensions allows us to build a non-rational monoid in which Kleene's theorem holds. Here again, the transferred ramification plays a key rôle and we prove that *an ideal extension of two rational semigroups verifies Kleene's theorem if and only if the transferred ramification and its inverse preserve rational sets.* Our construction is achieved with a slight adaptation of the reversal function which is the most standard non-rational function with this property (Example 3.4).

Section 4 finally deals with the problem mentioned above: the definition of a family of monoids that are "too simple" to be syntactic monoids of non-rational context-free languages. This is achieved by the definition of *rational thin monoids*: monoids that not only have an easy multiplication but also a very simple set of representatives. Their trace is roughly a finite union of copies of $\mathbb{N}$. We prove (Proposition 4.3) that the finitely generated monoids that are obtained from $\mathbb{N}$ by a finite sequence of Redei and ideal extensions by finite semigroups are all rational thin monoids. And the "monoïdes filiformes," that raised all the theory of rational monoids, are constructed in this way.

Section 1 is a brief survey of definitions and properties of rational semigroups, adapted from the ones of rational monoids.

## 1. Rational Semigroups

As it already appeared in the introduction the results concerning ideal extensions and Redei extensions are more likely stated with semigroups than with monoids. The definition of rational semigroup can of course be readily derived from the one of rational monoid. In order to make the present paper reasonably self-contained, we give, in this short first section, the definition of rational semigroups and express their properties that will be used here. These properties have all been stated and proved in Part I; the same proofs hold for rational semigroups.

If $X$ is a set, $X^+$ denotes the free semigroup generated by $X$ and $X^*$ the free monoid generated by $X$. The free monoid $X^*$ consists in the set of all sequences of finite length of elements of $X$, equipped with the concatenation. The identity element of $X^*$ is the empty sequence, denoted by $1_{X^*}$. The free semigroup $X^+$ is equal to $X^* \setminus 1_{X^*}$. The identity element of a monoid $M$ is denoted by $1_M$ and, for sake of simplicity, by 1 if there is no ambiguity.

For all definitions and properties of rational sets and rational relations the reader is referred to Eilenberg (1974), to Berstel (1979), or to Part I which contains a memento on this matter.

As for rational monoids, the definition of rational semigroups consists in a series of three definitions.

Definition 1.1. Let $S$ be a semigroup. A *generating system* of $S$ is a pair $(X, \alpha)$, where $X$ is a set and $\alpha$ is a surjective morphism from $X^+$ onto $S$.

Definition 1.2. A *trace* of a semigroup $S$ for a generating system $(X, \alpha)$ is a subset $R$ of $X^+$ such that $\alpha$ is a bijection from $R$ onto $S$. A *description* of $S$ for $(X, \alpha)$ is a mapping from $X^+$ into itself which associates with each element of $X^+$ its representative in a trace $R$.

DEFINITION 1.3. A semigroup is *rational* if it has a rational description for some generating system $(X, \alpha)$, i.e., a description which is a rational function from $X^+$ into $X^+$.

Since a monoid is also a semigroup, it is necessary to make the connection between this series of definitions and the one given in Part I for rational monoids. First, a monoid $M$ has two kinds of generating systems:

(a) if $M$ is considered as a monoid (i.e., $M$ is an object in the category of monoids), a generating system $(X, \alpha)$ of $M$ is such that $\alpha$ is a surjective morphism from $X^*$ onto $M$ and $1_M$ is the image of $1_{X^*}$. (In some sense, we can consider that every monoid shares the identity element.)

(b) if $M$ is considered as a semigroup (i.e., $M$ is an object in the category of semigroups), a generating system $(X, \alpha)$ of $M$ is such that $\alpha$ is a surjective morphism from $X^+$ onto $M$ and $1_M$ is the image of a non-empty word. (In this case, the identity elements of two distinct monoids are then distinct.)

Second, the rational functions from $X^+$ into $X^+$ are exactly the rational functions from $X^*$ into $X^*$, the domain and image of which are subsets of $X^+$ (cf. Eilenberg, 1974). From this and from the fact that one freely adds and substracts one element from the graph of a rational function, it follows that a rational monoid is a rational semigroup and that a rational semigroup, which is a monoid, is a rational monoid.

Let us now turn to the announced series of properties:

(P1) A rational semigroup has rational descriptions for every generating system. (This means that rationality is an intrinsic property for a semigroup, and does not depend on the choosen generating system in Definition 1.3—cf. Proposition I.1.2, that is, Proposition 2.1 of Part I.)

(P2) A rational semigroup is finitely generated (Proposition I.3.1).

(P3) A finitely generated subsemigroup of a rational semigroup is a rational semigroup (Theorem I.6.1).

(P4) Let $S$ be a rational semigroup and $\tau$ be a rational relation from $S$ into a semigroup $T$. Then the image of a rational set of $S$ by $\tau$ is a rational set of $T$ (Corollary I.4.4).

(P5) Let $S$ be a rational semigroup and let $T$ and $T'$ be two semigroups. Let $\tau: T \to S$ and $\tau': S \to T'$ be two rational relations. Then $\tau\tau'$ is a rational relation from $T$ into $T'$ (Corollary I.4.5).

Those properties will be referred to by means of their names (P$i$).

## 2. Redei Extensions

Redei extensions (the definition of which is recalled in Section 2.1) are a generalization of direct and semi-direct products. The purpose of this section is to characterize—with Theorem 2.3—those extensions of rational semigroups that are rational semigroups. A Redei extension is not necessarily finitely generated—and then not rational (P1). In order to overcome this difficulty, we define a restricted class of Redei extensions: the monoidal Redei extensions that produce only finitely generated semigroups. It is shown that monoidal extensions are, roughly speaking, described by right subsequential transducer (Section 2.3). It will be then easy to embed any Redei extension into a monoidal one (Section 2.4).

### 2.1. *Classical Theory of Redei Extensions*

It is a classical problem in group theory to construct the groups such that, given two groups $L$ and $H$, $H$ is a normal subgroup of $G$ and $L$ is isomorphic to the quotient of $G$ by $H$. Schreier's theory of extensions of groups solves the problem and gives a method to construct all groups $G$ with the prescribed properties (cf., for instance, Kurosch, 1974). The method is the following: the set $G$ is in bijection with the set $L \times H$ and the multiplication of the group $G$ is defined by two mappings: the first one from $L \times L$ into $H$ and the second one from $L$ into Aut $H$, the group of automorphisms of $H$. This construction has been generalized by Redei (1952). We briefly recall the basics of Redei's theory, with a slight difference: we consider semigroups instead of monoids.

Let $M$ and $F$ be two semigroups, let $\theta$ be a mapping from $F \times F$ into $M^1$ and $\psi$ be a mapping from $F$ into End $M^1$ ($M^1$ is equal to $M$ if $M$ is a monoid and to $M$ with an adjoint identity otherwise). We use Redei's notations (in fact, Kurosch's ones): we write $s_{f,g}$ for $(f, g)\theta$ and $m^f$ for $m(f\psi)$.

Let the law defined on the set $F \times M$ by

$$(*) \qquad \forall f, g \in F, \forall m, n \in M \qquad (f, m)(g, n) = (fg, s_{f,g} m^g n).$$

This law is associative if the following hold:

$$\text{(Red1)} \qquad \forall f, g \in F, \forall m \in M \qquad m^{fg} s_{f,g} = s_{f,g} (m^f)^g$$

$$\text{(Red2)} \qquad \forall f, g, h \in F \qquad s_{fg,h} (s_{f,g})^h = s_{f,gh} s_{g,h}.$$

DEFINITION 2.1. A semigroup $U$ is called a *Redei extension of $M$ by $F$* if there exist two mappings $\theta$ and $\psi$ which satisfy (Red1) and (Red2) and such that $U$ is isomorphic to the set $F \times M$ equipped with the multiplication:

$$\forall f, g \in F, \forall m, n \in M \qquad (f, m)(g, n) = (fg, s_{f,g} m^g n)$$

Such a Redei extension of $M$ by $F$ is denoted by $F \circ_{\theta,\psi} M$ or $F \circ M$ for short. As for Schreier's extensions, $(F \times F)\theta$ is called the *system of factors* and $F\psi$ is called the *system of endomorphisms* of the extension.

A series of examples will illustrate this definition.

EXAMPLE 2.1. If both $\theta$ and $\psi$ are trivial mappings (i.e., $(F \times F)\theta = \{1_{M^1}\}$ and $F\psi = \{\mathrm{id}_{M^1}\}$), then the conditions (Red1) and (Red2) are satisfied and the Redei extension $F \circ M$ is equal to the direct product $F \times M$.

EXAMPLE 2.2. If $\theta$ is trivial and if $\psi$ is a morphism, (Red1) and (Red2) are satisfied, and the Redei extension $F \circ M$ is equal to the semi-direct product $F \times_{\psi} M$.

EXAMPLE 2.3. Let $M = a^*$ and $F = \{1\}$. Let $\psi$ be the trivial mapping and let $s_{1,1} = a$, (Red1) and (Red2) are satisfied and the multiplication of $F \circ M$ is

$$\forall k, l \in \mathbf{N}, \qquad (1, a^k)(1, a^l) = (1, a^{k+l+1}).$$

EXAMPLE 2.4. Let $M = a^*$ and $F = \{1\}$. Let $\theta$ and $\psi$ be the mappings defined by

$$s_{1,1} = 1_M$$
$$m^1 = 1_M \qquad \forall m \in M.$$

The conditions (Red1) and (Red2) are satisfied and the multiplication of $F \circ M$ is

$$(1, a^k)(1, a^l) = (1, a^l).$$

EXAMPLE 2.5 (Perrot, 1977). Let $M = a^*$, $p$ an integer and $F = \mathbf{Z}/p\mathbf{Z}$. Let $q$ be a fixed integer. A trivial system of endomorphisms and the system of factors defined by

$$t_{f,g} = \begin{cases} 0 & \text{if } f + g \leqslant p - 1 \\ q & \text{if } f + g \geqslant p \end{cases}$$

(where the addition is taken in $\mathbf{Z}$) and

$$s_{f,g} = a^{t_{f,g}}$$

define a Redei extension (denoted by $M_{p,q}$), the multiplication of which is

$$\forall f, g \in [0, p-1], \qquad \forall k, l \in \mathbf{N}$$
$$(f, a^k)(g, a^l) = (f + g \bmod p, a^{k+l+t_{f,g}}).$$

Example 2.3 shows that a Redei extension of two monoids is not necessarily a monoid. Since we are interested by closure properties, we consider thus Redei extensions of *semigroups*.

The extension $F \circ M$ in Example 2.4 is not a finitely generated monoid since a minimal set of generators is $(1, a^*)$. Redei extensions of *semigroups* give rise to even simpler examples of non finitely generated extensions. Take, for instance, the direct product of $a^*$ by the finite semigroup $F = \{0, u\}$, with $u^2 = 0 \cdot u = u \cdot 0 = 0^2 = 0$, by $a^*$.

Example 2.4 illustrates also the fact that the definition of the multiplication of a Redei extension (by: $(f, m)(g, n) = (fg, s_{f,g} m^g n))$ is not symmetrical in $m$ and $n$. The Redei extensions we have defined are, in fact, *left* Redei extensions. We can also define *right* Redei extensions: as for (left) Redei extensions, let $\theta: F \times F \to M^1$, $\psi: F \to \operatorname{End} M^1$, $s_{f,g} = (f, g)\theta$, and $m^f = m(f\psi)$. Assume that $\theta$ and $\psi$ satisfy

$$\forall f, g \in F, \forall m \in M \qquad s_{f,g} m^{fg} = (m^g)^f s_{f,g}$$

$$\forall f, g, h \in F \qquad (s_{g,h})^f s_{f,gh} = s_{f,g} s_{fg,h}.$$

The set $M \times F$, equipped with the multiplication

$$(m, f)(n, g) = (mn^f s_{f,g}, fg),$$

is a semigroup and is called a right extension of $M$ by $F$.

The Redei extension of Example 2.4 is a left Redei extension which is not a right Redei extension. In what follows, we study only *left extensions*, called *extensions* for short. It is straightforward that dual properties of left extensions hold for right extensions. Remark that it is not necessary to define left and right extensions for groups, since we may equally consider a Schreier's extension as an union of left cosets or as an union of right cosets.

## 2.2. *Monoidal Redei Extensions and Their Descriptions*

Rational semigroups are not closed under Redei extension for such extensions are not necessarily finitely generated. We first restrict ourself to finitely generated extensions with the definition of monoidal Redei extensions.

DEFINITION 2.2. Let $F$ and $M$ be two monoids and $F \circ M$ a Redei extension of $M$ by $F$. $F \circ M$ is called a *monoidal Redei extension* if the system of factors and the system of endomorphisms are such that:

(i) $\forall f \in F$, $s_{f,1_F} = s_{1_F,f} = 1_M$

(ii) $\forall m \in M$, $m^{1_F} = m$.

From Definition 2.2 it follows that a monoidal Redei extension $F \circ M$ is a monoid, the identity of which is $(1_F, 1_M)$.

### 2.2.1. *A Generating System of $F \circ M$*

LEMMA 2.1. *Any monoidal Redei extension of a finitely generated monoid $M$ by a finite monoid $F$ is finitely generated.*

*Proof.* From the definition of monoidal Redei extension, it follows:

$$\forall m, n \in M, \qquad (1, m)(1, n) = (1, mn)$$

$$\forall f \in F, \forall m \in M, \qquad (f, 1)(1, m) = (f, m).$$

These two equalities imply that, if $\mathscr{G}$ is a set of generators of $M$, then $(F, 1) \cup (1, \mathscr{G})$ is a set of generators of $F \circ M$ (we denote by 1 both the identities of $M$ and $F$). ∎

The proof of Lemma 2.1 may be expressed in terms of generating systems for $M$ and $F \circ M$.

LEMMA 2.2. *Let $F \circ M$ be a monoidal Redei extension of a finitely generated monoid $M$ by a finite monoid $F$. Let $(X, \alpha)$ be a finite generating system of $M$ and let $\gamma$ be the morphism from $(X \cup F)^*$ into $F \circ M$ defined by*

$$\forall x \in X, \qquad x\gamma = (1, x\alpha)$$

$$\forall f \in F, \qquad f\gamma = (f, 1).$$

*Then $(X \cup F, \gamma)$ is a generating system of $F \circ M$.* ∎

### 2.2.2. *A Description of $F \circ M$*

Let $M$ be a finitely generated monoid, $(X, \alpha)$ a finite generating system of $M$ and $\beta$ a description of $M$ for $(X, \alpha)$. Let $F$ be a finite monoid. Lemma 2.2 gives a generating system $(X \cup F, \gamma)$ of a monoidal Redei extension of $M$ by $F$. Every word $u$ of $(X \cup F)^*$ has an unique representative in $F(X^*\beta)$. This defines a description of $F \circ M$ which will be denoted by $\eta$. The description $\eta$ is more easily understood as the composition of *two* mappings. The first one, $\delta$ maps $(X \cup F)^*$ onto $FX^*$ and may be written as

$$w\delta = (w\delta_1)(w\delta_2).$$

The second one is the description $\beta$. The description $\eta$ is then defined by

$$w\eta = (w\delta_1)(w\delta_2\beta).$$

The purpose of this section is to describe more precisely the mappings $\delta_1$ and $\delta_2$. Let $\zeta$ be the canonical morphism from the free monoid $F^*$ onto the monoid $F$ and $\pi_F$ the projection of $(X \cup F)^*$ onto $F^*$. Then $\delta_1 = \pi_F \zeta$. The extension $F \circ M$ is defined by a system of factors and a system of endomorphisms. We take, in $(X \cup F)^*$, sets of representatives of these two systems.

For all $f$, $g$ in $F$, let $u_{f,g}$ be a representative of $s_{f,g}$ in $X^*$, for instance,

$$u_{f,g} = s_{f,g}\alpha^{-1}\beta.$$

For all $x$ in $X$, for all $f$ in $F$, let $x^f$ be a representative of $(x\alpha)^f$ in $X^*$, for instance,

$$x^f = (x\alpha)^f\alpha^{-1}\beta.$$

Let $w \in (X \cup F)^*$, $(g, n) \in F \circ M$, such that $\omega\gamma = (g, n)$, $x \in X$ and $f \in F$. The multiplication in the extension $F \circ M$ leads to the following definition of $\delta_2$:

$$1\delta_2 = 1$$

$$(xw)\delta_2 = x^{w\delta_1} w\delta_2, \qquad \text{since} \quad (1, m)(g, n) = (g, m^g n) \text{ in } F \circ M$$

$$(fw)\delta_2 = u_{f,w\delta_1}(w\delta_2), \qquad \text{since} \quad (f, 1)(g, n) = (fg, s_{f,g}n), \text{ in } F \circ M.$$

LEMMA 2.3. *Let $\delta_1$ from $(X \cup F)^*$ into $F$, $\delta_2$ from $(X \cup F)^*$ into $X^*$ and $\eta$ defined as above. Then, for all $w$ in $(X \cup F)^*$,*

$$w\eta = (w\delta_1)(w\delta_2).$$

*Proof.* The proof is by induction on the length of $w$. Since $F(X^*\beta)$ is a trace of $F \circ M$ for $(X \cup F, \gamma)$, we have just to prove that $w\gamma = (w\delta_1, w\delta_2\alpha)$. This equality is satisfied by $w = 1$. Assume that $w\gamma = (w\delta_1, w\delta_2\alpha)$. Let $x$ be in $X$:

$$\begin{aligned}(xw)\gamma &= (1, x\alpha)(w\delta_1, w\delta_2\alpha)\\ &= (w\delta_1, (x\alpha)^{w\delta_1}(w\delta_2\alpha))\\ &= ((xw)\,\delta_1, (x^{w\delta_1}(w\delta_2))\alpha)\\ &= ((xw)\,\delta_1, (xw)\,\delta_2\alpha).\end{aligned}$$

Let $f$ be in $F$:

$$\begin{aligned}(fw)\gamma &= (f, 1)(w\delta_1, w\delta_2\alpha)\\ &= (f(w\delta_1), s_{f,w\delta_1}(w\delta_2\alpha))\\ &= ((fw)\,\delta_1, (u_{f,w\delta_1}(w\delta_2))\alpha)\\ &= ((fw)\,\delta_1, (fw)\,\delta_2\alpha). \quad \blacksquare\end{aligned}$$

### 2.2.3. *Representation of Monoïdal Redei Extensions by Right Subsequential Transducers*

The function $\delta$ defined above is not a description of $F \circ M$. But the knowledge of $\delta$ is sufficient to compute $\eta$, description of $F \circ M$, once the description $\beta$ of $M$ is known. The function $\delta$ describes indeed the mechanism of Redei extension, the multiplication in $M$ apart. In this section, it is shown, via $\delta$, that this mechanism of Redei extension can be realized as the *output function* of a right subsequential transducer.

In Part I, it has been proved, by algebraic means, that the direct product of a finite monoid by a rational monoid is a rational monoid. The proof of the same property using transducers will introduce the general method. A (left) subsequential transducer (see Berstel, 1979) is a generalized sequential machine—gsm—(see Eilenberg, 1974) together with an "exit" function from the set of states of the machine into the set of output words. The output of the subsequential tranducer on a word $f$ is the concatenation of the output function of the gsm on $f$ and of the value of the exit function on the state reached by the machine after the reading of $f$. A right subsequential transducer is a subsequential transducer that reads input words and writes output words from right to left.

With the above notations, $(X \cup F, \gamma)$ is a generating system of the direct product $F \times M$. In the case of direct product, the function $\delta_2$ is the projection on $X^*$. A transducer that realizes $\delta$ is obtained from left regular representation of $F$ over itself as the underlying automaton, while the output function is the identity on $X$ and is mute on $F$. More precisely, with the notations of Berstel (1979), let

$$\mathbf{T} = \langle X \cup F, X \cup F, F, 1_F, \rho \rangle$$

be the right subsequential with $X \cup F$ as input and output alphabet, $F$ as set of states and $1_F$ as initial state. The next-state function of $\mathbf{T}$ is

$$\begin{aligned} &(X \cup F) \times F \to F \\ &\qquad (x, f) \mapsto x \cdot f = f \\ &\qquad (g, f) \mapsto g \cdot f = gf, \end{aligned}$$

the output function of $\mathbf{T}$ is

$$\begin{aligned} &(X \cup F) \times F \to (X \cup F)^* \\ &\qquad (x, f) \mapsto x * f = x \\ &\qquad (g, f) \mapsto g * f = 1, \end{aligned}$$

and the exit function of **T** is

$$F \to (X \cup F)^*$$
$$f \mapsto f\rho = f.$$

Recall that these functions are extended to $(X \cup F)^* \times F$ by

$$1 \cdot f = f, \qquad (zw) \cdot f = z \cdot (w \cdot f)$$
$$1 * f = 1, \qquad (zw) * f = (z * (w \cdot f))(w * f)$$

and that the function $|\mathbf{T}|$ realized by **T** is defined by

$$(w)\,|\mathbf{T}| = (w \cdot 1_F)\,\rho(w * 1_F).$$

It is straightforward that, with the next-state, output, and exit functions we have chosen, we obtain

$$w * 1_F = w\pi_X$$
$$(w \cdot 1_F)\rho = w\pi_F.$$

The case of general monoidal Redei extensions is a slight generalization of this construction.

LEMMA 2.4. *Let $F \circ M$ be a monoidal Redei extension and $\delta$ be the mapping associated with this extension, then $\delta$ is realized by a right subsequential transducer.*

*Proof.* Let

$$\mathbf{T} = \langle X \cup F, X \cup F, F, 1_F, \rho \rangle$$

be the right subsequential transducer with $X \cup F$ as input and output alphabet, $F$ as set of states and $1_F$ as initial state as above. The next-state function of **T** is

$$(X \cup F) \times F \to F$$
$$(x, f) \mapsto x \cdot f = f$$
$$(g, f) \mapsto g \cdot f = gf$$

and the exit function of **T** is

$$F \to (X \cup F)^*$$
$$f \mapsto f\rho = f$$

as above.

The output function of $\mathbf{T}$ is

$$(X \cup F) \times F \to (X \cup F)^*$$
$$(x, f) \mapsto x * f = x^f$$
$$(g, f) \mapsto g * f = u_{g,f}.$$

The function $|\mathbf{T}|$ is defined by $(w)|\mathbf{T}| = (w \cdot 1_F)(w * 1_F)$.

Intuitively, after reading (from right to left) a word $w$, the transducer is in the state $w\delta_1$; if it reads a letter $x$ of $X$, it writes $x^{(w\delta_1)}$ (just like $\delta_2$) and stays in the state $w\delta_1 = (xw)\delta_1$; if it reads a letter $f$ of $F$, it writes $u_{f,w\delta_1}$ (just like $\delta_2$) and goes in the state $f(\omega\delta_1) = (fw)\delta_1$.

More precisely, we prove, by induction on $|w|$ that $w \cdot 1_F = w\delta_1$ and $w * 1_F = w\delta_2$. These two equalities are satisfied by $w = 1$. Assume they are satisfied by $w$. Let $x$ be in $X$ and $f$ be in $F$:

$$(xw) \cdot 1_F = x \cdot (w\delta_1) = w\delta_1 = (xw)\delta_1$$
$$(fw) \cdot 1_F = f \cdot (w\delta_1) = f(w\delta_1) = (fw)\delta_1$$
$$\begin{aligned}(xw) * 1_F &= [x * (w \cdot 1_F)](w * 1_F)\\ &= [x * (w\delta_1)](w\delta_2)\\ &= x^{w\delta_1}(w\delta_2)\\ &= (xw)\delta_2\\ (fw) * 1_F &= [f * (w\delta_1)](w\delta_2)\\ &= u_{f,w\delta_1}(w\delta_2)\\ &= (fw)\delta_2.\end{aligned}$$

Thus $\mathbf{S}$ realizes $\delta$. ∎

The direct product could be realized by a left subsequential transducer as well as by a right subsequential transducer. The definition of the multiplication of a (left) Redei extension with non-trivial systems of factors and of endomorphisms makes it necessary to use a right subsequential transducer and a right Redei extension will be realized by a left subsequential transducer.

### 2.2.4. *Monoidal Redei Extensions of Rational Monoids by Finite Monoids*

We keep the notations of the previous paragraphs. Since the function realized by a right subsequential transducer is rational, (cf. Berstel, 1979), it is an immediate consequence of the previous lemma that the rationality of $\eta$ depends only upon the rationality of $\beta$ and thus:

THEOREM 2.1. *Any monoidal Redei extension of a rational monoid by a finite monoid is a rational monoid.*

Since a semi-direct product is a monoidal Redei extension, we have

COROLLARY 2.1. *Any semi-direct product of a rational monoid by a finite monoid is a rational monoid.*

We note also that the extensions presented in the Example 2.5 are monoidal and the monoids $M_{p,q}$ are rational.

## 2.3. *The General Case*

Although a Redei extension of a finitely generated semigroup by a finite semigroup is not necessarily finitely generated, its multiplication is always "easy" to describe.

The definition of generalized Redei extension is a construction that takes place in the category of semigroups and the monoidal Redei extension we have defined is the corresponding construction in the category of monoids. Since any object of the category of semigroups can be transported in the category of monoids by adding it an identity, what has been proved for monoidal Redei extension can be used for a general Redei extension.

THEOREM 2.2. *Any Redei extension of a rational semigroup by a finite semigroup is a rational ideal of a rational monoid.*

*Proof.* Let $M$ be a rational semigroup and $F$ be a finite semigroup. Let $M_1$ and $F_1$ be the monoids obtained by adding identities to $M$ and $F$. We extend the systems of factors and of endomorphisms by

$$\begin{aligned} &\forall f \in F_1, && s_{1,f} = s_{f,1} = 1 \\ &\forall f \in F_1, && 1^f = 1 \\ &\forall m \in M_1, && m^1 = m. \end{aligned}$$

We obtain a monoidal Redei extension $F_1 \circ M_1$. $F \circ M$ is an ideal of $F_1 \circ M_1$. Let $(X, \alpha)$ be a finite generating system of the semigroup $M$, i.e., $M = X^+\alpha$. Let $\beta$ be a rational description of $M$ for $(X, \alpha)$. We extend $\alpha$ in a surjective morphism from $X^*$ onto $M_1$ $(1\alpha = 1)$ and $\beta$ in a description of $M_1$ for $(X, \alpha)$ $(1\beta = 1)$. Then $\beta$ is a rational description of $M_1$. By Theorem 2.1, the monoidal Redei extension $F_1 \circ M_1$ is a rational monoid. Let $\gamma$ the surjective morphism from $(X \cup F_1)^*$ onto $F_1 \circ M_1$, defined in Lemma 2.2. Then $F \circ M = (FX^+)\gamma$. Since $FX^+ \in \mathrm{Rat}(X \cup F_1)^*$, $F \circ M$ is a rational subset of $F_1 \circ M_1$. ∎

The following theorem characterizes all the Redei extensions (of a rational semigroup by a finite semigroup) which are rational. Theorem 2.2 and property (P3) imply:

THEOREM 2.3. *A Redei extension of a rational semigroup by a finite semigroup is a rational semigroup if and only if it is finitely generated.*

## 3. IDEAL EXTENSIONS

The ideal extension is the construction inverse to the Rees quotient (Definitions 3.1 and 3.2). Petrich (1973) gave a standard construction of ideal extensions, by means of two functions: a representation by bitranslations (Definitions 3.3 to 3.5) and a ramification (Definition 3.6). This construction is recalled in Section 3.1. In order to give the description of an ideal extension, we transfer the representation and the ramification in the free monoid (Section 3.2). It will then appear that the rationality of the ideal extension only depends upon the rationality of the transferred ramification (Section 3.3, Theorem 3.1). Corollaries 3.2 to 3.6 are particular cases of regular (and thus rational) ideal extensions.

In Section 3.4, the theory of ideal extensions is used to build a non-rational monoid in which Kleene's theorem holds: by Theorem 3.2, the problem comes down to exhibiting a transferred ramification which is not rational and which, as well as its inverse, maps rational sets into rational sets (Example 3.4).

### 3.1. *Classical Theory of Ideal Extensions*

Recall first the definitions of Rees quotient and ideal extension (cf. Clifford and Preston, 1961).

DEFINITION 3.1. Let $N$ be an ideal of a semigroup $U$. The *Rees quotient* of $U$ by $N$, denoted by $U//N$, is the quotient of $U$ by the following congruence:

$$\forall u, v \in U, \qquad u \equiv v \Leftrightarrow \begin{cases} u = v \text{ or} \\ u \in N \text{ and } v \in N. \end{cases}$$

We may describe $U//N$ as the result of collapsing $N$ into a single (zero) element, while the elements of $U$ outside $N$ retain their identity.

Let $M$ be a semigroup. We note by $M^0$ the semigroup equal to $M$ if $M$ has a zero and to $M$ with an adjoint zero otherwise; we note by $M^{\bullet}$ the set $M^0 \backslash 0_{M^0}$; that is, $M^{\bullet}$ equals $M$, if $M$ has no zero, and $M \backslash 0_M$, otherwise.

DEFINITION 3.2. A semigroup $U$ is an *ideal extension* of a semigroup $N$ by a semigroup $M$ if it contains $N$ as an ideal and if the Rees quotient $U//N$ is equal to $M^0$, up to isomorphisms (then $U\backslash N = M^{\bullet}$).

A natural way of building an ideal extension of $N$ by $M$ consists in providing the set $U = M^{\bullet} \cup N$ with a multiplication $*$ such that $N$ is an ideal of $U$ and $M^0$ is isomorphic to the Rees quotient $U//N$. Obviously, this multiplication has to coincide with this one of $M$ for the pairs $(a, b)$ of $M^{\bullet} \times M^{\bullet}$ such that $ab$ is in $M^{\bullet}$ and with this one of $N$ for the pairs $(a, b)$ of $N \times N$. It only remains to define the products:

(1) $a * b$ (and $b * a$) for $a$ in $M^{\bullet}$ and $b$ in $N$

(2) $a * b$ for $a, b$ in $M^{\bullet}$ such that $ab = 0$.

We define two functions (with values in $N$) which permit us to compute these two types of products and which satisfy certain properties intended to ensure the associativity of the multiplication in $U$. In doing this, we follow Petrich (1973). We first define the function which permits to multiply elements of $M^{\bullet}$ by elements of $N$.

### 3.1.1. *Representations by Bitranslations*

DEFINITION 3.3. Let $N$ be a semigroup. A *left-translation* of $N$ is a mapping $\lambda$ from $N$ into $N$ such that, for all $n, n'$ in $N$,

$$\lambda(nn') = (\lambda n)n'$$

(with our usual notations, we should have written $n\lambda$, but we prefer the notation $\lambda n$; see below).

A *right-translation* of $N$ is a mapping $\rho$ from $N$ in $N$ such that, for all $n, n'$ in $N$,

$$(nn')\rho = n(n'\rho).$$

A left-translation $\lambda$ and a right-translation $\rho$ are *linked* if, for all $n, n'$ in $N$,

$$n(\lambda n') = (n\rho)n'$$

and those translations are *permutable* if, for all $n$ in $N$,

$$(\lambda n)\rho = \lambda(n\rho).$$

DEFINITION 3.4. A pair of linked left and right-translations of $N$ is called a *bitranslation* of $N$.

A set $S$ of bitranslations of $N$ is *permutable* if, for all $(\lambda, \rho)$ and $(\lambda', \rho')$ in $S$, $\lambda'$ and $\rho$ are permutable.

An important particular case of left-translation (resp. right-translation) is the multiplication to the left (resp. to the right) by an element $n$ of $N$. Such a left-translation is called an *inner left-translation* (resp. *inner right-translation*) and denoted by $\lambda_n$ (resp. $\rho_n$).

(Remark that the equality $\lambda_n n' = nn'$ is more readable than $n'\lambda_n = nn'$ and this justifies that we depart from the normal notations for mappings.) For all $n$ in $N$, $\lambda_n$, and $\rho_n$ are linked: we note $\pi_n$ the bitranslation $(\lambda_n, \rho_n)$. For all $n, n'$ in $N$, $\lambda_n$, and $\rho_{n'}$ are permutable. Thus, the set of inner bitranslations is permutable.

We denote by $\Lambda(N)$ (resp. $P(N)$, $\Omega(N)$) the monoid of all left-translations (resp. right-translations, bitranslations) of $N$.

DEFINITION 3.5. Let $M$ and $N$ be two semigroups. A *representation* of $M^\bullet$ by left-translations (resp. right-translations, bitranslations) of $N$ is a mapping $\delta$ from $M^\bullet$ into $\Lambda(N)$ (resp. $P(N)$, $\Omega(N)$) such that:

$$\forall m, m' \in M^\bullet \qquad mm' \in M^\bullet \Rightarrow (mm')\delta = (m\delta)(m'\delta).$$

Note that, if $mm' = 0$ then $(mm')\delta$ is not defined, but $(m\delta)(m'\delta)$ does exist. We shall write $\delta^m$ instead of $m\delta$, which allows us to write $n\delta^m$ instead of $n(m)\delta$, and stresses the fact that $\delta^m$ is a mapping.

Let us now define the function which allows us to calculate the products $a * b$ for $a, b$ in $M^\bullet$ such that $ab = 0$.

### 3.1.2. *Ramification Function*

Let $W_M$ be the set $\{(m, m') \in M^\bullet \times M^\bullet \mid mm' = 0\}$.

DEFINITION 3.6. A *ramification function* of $M^\bullet$ into $N$ is a mapping $\psi$ from $W_M$ into $N$ which satisfies

$$(mm', m'')\psi = (m, m'm'')\psi$$

for all $m, m', m''$ in $M^\bullet$ such that $(mm', m'')$, $(m, m'm'')$ are in $W_M$.

A ramification function $\psi$ of $M^\bullet$ into $N$ is *adapted* to a representation $\theta$ of $M^\bullet$ by bitranslations of $N$ if the following conditions hold:

| | | |
|---|---|---|
| (C1) | $\theta^m\theta^{m'} = \pi_{(m,m')\psi}$ | if $(m, m') \in W_M$ |
| (C2) | $(mm', m'')\psi = \lambda^m((m', m'')\psi)$, | if $(mm', m''), (m', m'') \in W_M$ |
| (C3) | $(m, m'm'')\psi = ((m, m')\psi)\, \rho^{m''}$ | if $(m, m'm''), (m, m') \in W_M$ |
| (C4) | $\lambda^m((m', m'')\psi) = ((m, m')\psi)\rho^{m''}$, | if $(m', m''), (m, m') \in W_M$. |

Note that (C1) is a condensed notation for:

$$\forall (m, m') \in W_M, \qquad \forall n \in N,$$
$$\lambda^m \lambda^{m'} n = (m, m')\psi n$$
$$n \rho^m \rho^{m'} = n(m, m')\psi.$$

All the definitions and conditions have been designed in view of the following:

PROPOSITION 3.1. *Let $M$ and $N$ be two semigroups. Let $\theta$ be a representation of $M^\bullet$ by permutable bitranslations of $N$ and $\psi$ be a ramification function of $M^\bullet$ into $N$, adapted to $\theta$. Then set $U = M^\bullet \cup N$, equipped with the multiplication $*$ defined by*

$$a * b = \begin{cases} a\rho^b & \text{if } a \in N, b \in M^\bullet \\ \lambda^a b & \text{if } a \in M^\bullet\ b \in N \\ (a, b)\psi & \text{if } (a, b) \in W_M \\ ab & \text{otherwise} \end{cases}$$

*is an ideal extension of $N$ by $M$, denoted by $\langle M, N, \theta, \psi \rangle$. Conversely, any ideal extension of $N$ by $M$ can be constructed that way.*

*Remark* 3.1. When $M^\bullet$ is a semigroup, i.e., when $M$ has no zero or when $M$ has an adjoint zero, the domain of $\psi$ is empty and the extensions of $N$ by $M$ are completely determined by the representation $\theta$. In this case, the ideal extensions of $N$ by $M$ are the same as the ideal extensions of $N$ by $M^\bullet$. In view of this remark, we shall always suppose that either $M$ has no zero (and $M^\bullet = M$) or that $M^\bullet$ is not a semigroup.

*Remark* 3.2. If $N$ is a monoid, then every (left or right) translation is completely determined by its value on $1_N$:

$$\forall n \in N, \qquad \lambda n = \lambda(1_N n) = (\lambda 1_N)n.$$

This implies that $\Lambda(N) = P(N) = \Omega(N) = N$, i.e., every translation is inner. By (C1), $\theta$ determines $\psi$ and thus the extension of $N$ by $M$.

*Remark* 3.3. If $N$ is a free semigroup $Y^+$ and if $\psi$ is a ramification adapted to a representation $\theta$, then, for each $(m, m')$ in $W_M$, $(m, m')\psi$ is the unique word $w$ of $Y^*$ such that

$$\forall y \in Y, \qquad \lambda^m(\lambda^{m'} y) = wy, \qquad (y\rho^m)\rho^{m'} = yw.$$

Thus $(m, m')\psi = [\lambda^m(\lambda^{m'} y)]\, y^{-1}$ for each $y$ in $Y$.

3.1.3. *Extension of the Representation by Bitranslations*

Let $U = \langle M, N, \theta, \psi \rangle$ be an ideal extension of $N$ by $M$, and $\theta = (\lambda, \rho)$. By definition, the representation $\lambda$ of $M^{\bullet}$ verifies $\lambda^m \lambda^{m'} = \lambda^{mm'}$ for $m, m'$ in $M^{\bullet}$ such that $mm'$ is in $M^{\bullet}$ which means that $\lambda$ is almost a morphism from $M^{\bullet}$ into $\Lambda(N)$. In the following proposition, $\lambda$ and $\rho$ are extended to true morphisms from $U$ into $\Lambda(N)$ and $P(N)$, respectively.

PROPOSITION 3.2. *Let $U = \langle M, N, \theta, \psi \rangle$ an ideal extension of $N$ by $M$, with $\theta = (\lambda, \rho)$. For every $n$ in $N$, let $\lambda^n$ be equal to the inner left-translation $\lambda_n$ and $\rho^n$ equal to the inner right-translation $\rho_n$. Then $\lambda$ is a morphism from $U$ into $\Lambda(N)$, $\rho$ is a morphism from $U$ into $P(N)$, and $\theta = (\lambda, \rho)$ is a representation of $U$ by permutable bitranslations of $N$. Moreover, with this convention*

$$\forall a \in U, \forall n \in N, \qquad a * n = \lambda^a n, \qquad n * a = n\rho^a.$$

*Proof.* We prove that $\lambda^{a*b} = \lambda^a \lambda^b$ for all $a, b$ in $U$. This is already known when $a$ and $b$ are in $N$ or when $a$, $b$, and $ab$ are in $M^{\bullet}$.

If $a \in N$, $b \in M^{\bullet}$ and $n \in N$:

$$\lambda^{a*b} n = (a\rho^b)n = a(\lambda^b n) = \lambda^a \lambda^b n.$$

If $a \in M^{\bullet}$, $b \in N$, and $n \in N$:

$$\lambda^{a*b} n = (\lambda^a b)n = \lambda^a (bn) = \lambda^a \lambda^b n.$$

If $(a, b) \in W_M$, i.e., $a, b$ are in $M^{\bullet}$ and $ab = 0$, then

$$\lambda^a \lambda^b = \lambda_{(a,b)\psi} \qquad \text{by (C1)}$$

and by definition $a * b = (a, b)\psi$; hence our convention gives $\lambda^a \lambda^b = \lambda^{a*b}$. Thus $\lambda$ is a morphism from $U$ into $\Lambda(N)$ and, in the same way, $\rho$ is a morphism from $U$ into $P(N)$.

It is clear that $\lambda^a$ and $\rho^a$ are linked for all $a$ in $U$. If $a, b \in N$ or if $a, b \in M^{\bullet}$ $\lambda^a$ and $\rho^b$ are permutable. If $a \in M^{\bullet}$ $b \in N$, and $n \in N$,

$$(\lambda^a n)\rho^b = (\lambda^a n)b = \lambda^a(nb) = \lambda^a(n\rho^b);$$

that is $\lambda^a$ and $\rho^b$ are permutable. The case $a \in N$ and $b \in M^{\bullet}$ is similar. Thus $\theta$ is a representation of $U$ by permutable bitranslations of $N$. The last two equalities of the proposition are straightforward. ∎

Proposition 3.2 replaces the definition of a multiplication in $U$ by the definition of a representation of $U$ (which obviously implies the existence of the multiplication of $U$). This slight change of point of view will bring great simplifications in the definition of the canonical description of $U$ in the next section.

## 3.2. *Description of Ideal Extensions*

Let $M$ and $N$ be two semigroups and let $U = \langle M, N, \theta, \psi \rangle$ an ideal extension of $N$ by $M$. We shall keep in the sequel the following notations: let $(X, \alpha)$ be a generating system of $M$ (i.e., $\alpha$ is a surjective morphism from $X^+$ onto $M$), and $(Y, \zeta)$, and $(Y, \zeta)$ a generating system of $N$. Let $\beta$ be a description of $M$ for $(X, \alpha)$ and $\xi$ be a description of $N$ for $(Y, \zeta)$.

We denote by $I$ the ideal $0_M \alpha^{-1}$ of $X^*$. We can assume that $X$ and $I$ are disjoint: by Remark 3.1, either $M$ has no zero and $I$ is empty, or $M^\bullet$ is not a semigroup and $(X \backslash I, \alpha)$ is a generating system of $M$. Under this assumption, the application $\gamma$, defined by

$$\forall x \in X, \qquad x\gamma = x\alpha$$
$$\forall y \in Y, \qquad y\gamma = y\zeta,$$

maps $X \cup Y$ into $U$ and can be extended into a surjective morphism from $(X \cup Y)^+$ onto $U$.

A description of $U$ for $(X \cup Y, \gamma)$ is a mapping from $(X \cup Y)^+$ into itself which has the same mapping equivalence than $\gamma$. Let $f$ be a word of $(X \cup Y)^+$. If $f$ is in $X^+ \backslash I$, then $f\gamma = f\alpha$ and we can choose $f\beta$ as a representative of $f$. If $f$ is not in $X^+ \backslash I$, then $f\gamma$ is in $N$ and we can choose the representative of this element of $N$ in $Y^+$, i.e., $f\gamma\zeta^{-1}\xi$. This can be summed up in the following proposition:

PROPOSITION 3.3. *The mapping $\eta$ from $(X \cup Y)^+$ into $(X \cup Y)^+$ defined by*

$$f\eta = \begin{cases} f\beta, & \textit{if } f \in X^+ \backslash I \\ f\gamma\zeta^{-1}\xi, & \textit{otherwise} \end{cases}$$

*is a description of $U$ for $(X \cup Y, \gamma)$.*

This expression for $\eta$ is not satisfactory since it contains $\gamma$. On the contrary, we want $\eta$ to replace $\gamma$ and the multiplication in $U = \langle M, N, \theta, \psi \rangle$. In order to define $\eta$ purely in terms of mappings from $(X \cup Y)^+$ into itself, we associate with the function $\theta = (\lambda, \rho)$ a function $\bar{\theta} = (\bar{\lambda}, \bar{\rho})$, where $\bar{\lambda}$ and $\bar{\rho}$ are representations of $(X \cup Y)^*$ by (left and right) translations of $Y^+$, and we associate with the ramification $\psi$ a function $\bar{\psi}$ from $I$ into $Y^+$.

### 3.2.1. *The Function $\bar{\theta}$*

First note that a left-translation of $Y^+$ is completely determined by its values on each letter $y$ of $Y$ and that a representation of $X^+$ by left-translations of $Y^+$ is completely determined by its values on each letter $x$ of $X$ and each letter $y$ of $Y$.

LEMMA 3.1. *Let $\lambda$ be a representation of $M^{\bullet}$ by left-translations of $N$. For every pair $(x, y)$ with $x$ in $X$ and $y$ in $Y$, let $w_{x,y}$ be the representative, for the description $\xi$, of the image by $\lambda^{x\alpha}$ of the element $y\zeta$; i.e., $w_{x,y} = (\lambda^{x\alpha}(y\zeta))\,\zeta^{-1}\xi$.*

*Let $\bar{\lambda}$ be the representation of $X^+$ by left-translations of $Y^+$ defined by*

$$\forall x \in X, \forall y \in Y, \qquad \bar{\lambda}^{x} y = w_{x,y}.$$

*Then, the following holds*:

$$\text{(L)} \quad \forall x_1, ..., x_n \in X, \forall v \in Y^+, (\bar{\lambda}^{x_1 \cdots x_n} v)\zeta = \lambda^{x_n\alpha}(v\zeta).$$

*Proof.* From the definition of $w_{x,y}$, $(\bar{\lambda}^{x} y)\zeta = \lambda^{x\alpha}(y\zeta)$. Since $\bar{\lambda}^{x}$ and $\lambda^{x\alpha}$ are left-translations and since $\zeta$ is a morphism, $(\bar{\lambda}^{x} v)\zeta = \lambda^{x\alpha}(v\zeta)$ for all $x$ in $X$ and $v$ in $Y^+$. The verification of the property follows from the induction formula:

$$\begin{aligned} &\forall x_1, ..., x_{n+1} \in X, \qquad \forall v \in Y^+, \\ (\bar{\lambda}^{x_1 \cdots x_n x_{n+1}} v)\zeta &= (\bar{\lambda}^{x_1 \cdots x_n}(\bar{\lambda}^{x}_{n+1} v))\zeta \\ &= \lambda^{x_1\alpha} \cdots \lambda^{x_n\alpha}((\bar{\lambda}^{x_{n+1}} v)\zeta) \\ &= \lambda^{x_1\alpha} \cdots \lambda^{x_n\alpha}\lambda^{x_{n+1}\alpha}(v\zeta). \quad \blacksquare \end{aligned}$$

With the convention that $\bar{\lambda}^{1} = \mathrm{id}_{Y^+}$ and $\lambda^{1\alpha} = \mathrm{id}_N$ the representation $\bar{\lambda}$ is extended to a representation of the whole monoid $X^*$ for which property (L) holds.

If $f = x_1 \cdots x_n$ is in $X^* \backslash I$, (L) may be written as $(\bar{\lambda}^{f} v)\zeta = \lambda^{f\alpha}(v\zeta)$: this compact writting is not possible for $f$ in $I$ (since $f\alpha = 0$ then). As we have extended $\lambda$ to a representation of $U$ (by left-translations of $N$) in Proposition 3.2, we extend $\bar{\lambda}$ to a representation of $(X \cup Y)^+$ (by left-translations of $Y^+$).

LEMMA 3.2. *For every $y$ in $Y$, let $\bar{\lambda}^{y} = \lambda_y$, the inner left-translation of $Y^+$ associated to $y$. $\bar{\lambda}$ is thus extended into a representation of $(X \cup Y)^*$ by left-translations of $Y^+$ and the following holds*:

$$\text{(L}'\text{)} \quad \forall f \in (X \cup Y)^*, \forall v \in Y^+, (\bar{\lambda}^{f} v)\gamma = \lambda^{f\gamma}(v\gamma).$$

*Proof.* By induction on the length of $f$. Let $f$ be in $(X \cup Y)^*$, $z$ in $X \cup Y$ and $v$ in $Y^+$:

$$(\bar{\lambda}^{fz} v)\gamma = \lambda^{f\gamma}((\bar{\lambda}^{z} v)\gamma) \qquad \text{by hypothesis of induction.}$$

If $z \in X$, then:

$$\begin{aligned}(\bar{\lambda}^{fz}v)\gamma &= \lambda^{f\gamma}((\bar{\lambda}^{z}v)\zeta) && \text{since } \bar{\lambda}^{z}v \in Y^{+}\\ &= \lambda^{f\gamma}(\lambda^{z}(v\zeta)) && \text{by (L)}\\ &= \lambda^{f\gamma}\lambda^{z\gamma}(v\gamma) && \text{since } z \in X \text{ and } v \in Y^{+}\\ &= \lambda^{(fz)\gamma}(v\gamma) && \text{by Proposition 3.2.}\end{aligned}$$

If $z \in Y$, then:

$$\begin{aligned}(\bar{\lambda}^{fz}v)\gamma &= \lambda^{f\gamma}((zv)\gamma)\\ &= \lambda^{f\gamma}((z\gamma)(v\gamma))\\ &= \lambda^{f\gamma}\lambda^{z\gamma}(v\gamma) && \text{since } z\gamma \in N \text{ and by definition of } \lambda^{n} \text{ for } n \in N\\ &= \lambda^{(fz)\gamma}(v\gamma) && \text{by Proposition 3.2.} \quad \blacksquare\end{aligned}$$

The representation $\rho$ of $U$ by right-translations of $N$ gives rise to a lemma that is dual to Lemma 3.2:

LEMMA 3.3. *Let $\bar{\rho}$ be the representation of $(X \cup Y)^*$ by right-translations of $Y^+$ associated to $\rho$; then*

(R′) $\forall f \in (X \cup Y)^*$, $\forall v \in Y^+$, $(v\bar{\rho}^{f})\gamma = (v\gamma)\rho^{f\gamma}$.

*Remark* 3.4. Although $\bar{\lambda}$ and $\bar{\rho}$ are representations of $(X \cup Y)^*$ by left- and right-translations of $Y^+$, $\bar{\theta}$ is not necessarily a representation of $(X \cup Y)^*$ by bitranslations of $Y^+$. This is shown by the following example:

EXAMPLE 3.1. Let $M$ be the semigroup:

$$M = \{a, b, 0\}, \qquad a^2 = a,\ b^2 = b,\ ab = ba = 0.$$

Let $X = \{a, b\}$ and let $\alpha$ be the obvious morphism ($a\alpha = a$ and $b\alpha = b$). The mapping $\beta$, from $X^+$ into $X^+$, defined by $a^n\beta = a$, $b^n\beta = b$, $(uabv)\beta = (ubav)\beta = ab$, is a natural description of $M$.

Let $N = \{x, y, 0\}$, $Y = \{x, y\}$, $\zeta$ and $\xi$ some copies of $M$, $X$, $\alpha$, and $\beta$. Recall that $\pi_x = (\lambda_x, \rho_x)$ is the inner bitranslation by $x$. Let $\theta$ be the representation of $M^{\cdot}$ by bitranslations of $N$ defined by

$$\theta^a = \pi_x \qquad \text{and} \qquad \theta^b = \pi_y.$$

We obtain

$$y(\bar{\lambda}^{a}y) = y(xy)\zeta^{-1}\xi) = yxy$$
$$(y\bar{\rho}^{a})y = ((yx)\zeta^{-1}\xi)\,y = xyy.$$

Thus $\bar{\lambda}^a$ and $\bar{\rho}^a$ are not linked and $\bar{\theta}^a$ is not a bitanslation.

However, the following lemma expresses that, modulo $\zeta$, $\bar{\theta}$ is a representation of $(X \cup Y)^*$ by permutable bitranslations of $Y^+$.

LEMMA 3.4. $(\mathrm{T}_1)$ $\forall f \in (X \cup Y)^*, \forall v, w \in Y^+, (v(\bar{\lambda}^f w))\zeta = ((v\bar{\rho}^f)w)\zeta$.
$(\mathrm{T}_2)$ $\forall f, g \in (X \cup Y)^*, \forall v \in Y^+, ((\bar{\lambda}^f v)\,\bar{\rho}^g)\zeta = (\bar{\lambda}^f(v\bar{\rho}^g))\zeta$.

$(\mathrm{T}_1)$ means that $\bar{\lambda}^f$ and $\bar{\rho}^f$ are linked modulo $\zeta$. $(\mathrm{T}_2)$ means that $\bar{\lambda}^f$ and $\bar{\rho}^g$ are permutable modulo $\zeta$. The verification of Lemma 3.4 is a straightforward computation using (L′), (R′), and the fact that $\lambda^{x\alpha}$ and $\rho^{x\alpha}$ are linked for all $x$ in $X$ and that $\lambda^{x\alpha}$ and $\rho^{y\alpha}$ are permutable for all $x$, $y$ in $X$. $(\mathrm{T}_2)$ allows us to write $(\bar{\lambda}^f v\bar{\rho}^g)\zeta$ and $(\bar{\lambda}^f v\bar{\rho}^g)\xi$.

The following lemma gives the restriction of $\eta$ to $(X \cup Y)^* Y (X \cup Y)^*$.

LEMMA 3.5. *Let $f = gyh$ be an element of $(X \cup Y)^* Y (X \cup Y)^*$, with $y$ in $Y$, then*

$$f\eta = (\bar{\lambda}^g y \bar{\rho}^h)\xi.$$

*Proof.* Let $g, h$ in $(X \cup Y)^*$ and $y$ in $Y$:

$$\begin{aligned}(\bar{\lambda}^g y\bar{\rho}^h)\zeta &= (\bar{\lambda}^g y\bar{\rho}^h)\gamma \\ &= \lambda^{g\gamma}(y\gamma)\,\rho^{h\gamma} && \text{by (L′) and (R′)} \\ &= g\gamma * y\gamma * h\gamma && \text{by Proposition 3.2} \\ &= (gyh)\gamma.\end{aligned}$$

Thus $(\bar{\lambda}^g y\bar{\rho}^h)\xi = (gyh)\,\gamma\zeta^{-1}\xi = (gyh)\eta$. ∎

3.2.2. *The Function $\bar{\psi}$*

The multiplication in $U$ involves a ramification function $\psi$. In order to describe this multiplication, it is necessary to associate a function $\bar{\psi}$ with the function $\psi$. Since the domain of $\psi$ is $W_M = \{(m, m') \in M^{\bullet} \times M^{\bullet} \mid mm' = 0\}$, it is natural to define the function $\bar{\psi}$ on the set:

$$V = \{g \in I \mid \exists g_1, g_2 \in X^+ \setminus I,\ g = g_1 g_2\}.$$

Note that $V = \{g_1 g_2 \mid (g_1\alpha, g_2\alpha) \in W_M\}$.

LEMMA 3.6. *For every $g$ in $V$, the element $(g_1\alpha, g_2\alpha)\,\psi\zeta^{-1}\xi$ does not depend upon the factorization $g = g_1 g_2$ such that $g_1, g_2$ are in $X^+ \setminus I$. Let $\bar{\psi}$ be the mapping from $V$ into $Y^+$ defined by*

$$g\bar{\psi} = (g_1\alpha, g_2\alpha)\,\psi\zeta^{-1}\xi \qquad \text{with } g = g_1 g_2;\ g_1, g_2 \in X^+ \setminus I.$$

*Then $\bar{\psi}$ is equal to the restriction of $\eta$ to $V$.*

*Proof.* Let $g \in V$ and $g_1, g_2 \in X^+ \backslash I$ such that $g = g_1 g_2$:

$$\begin{aligned}(g_1\alpha, g_2\alpha)\psi\zeta^{-1}\xi\zeta &= (g_1\alpha, g_2\alpha)\psi \\ &= g_1\alpha * g_2\alpha && \text{by definition of the multiplication } * \\ &= g_1\gamma * g_2\gamma && \text{since } g_1, g_2 \in X^+\backslash I \\ &= (g_1 g_2)\gamma \\ &= g\gamma.\end{aligned}$$

Hence,

$$(g_1\alpha, g_2\alpha)\psi\zeta^{-1}\xi = g\gamma\zeta^{-1}\xi = g\eta. \quad \blacksquare$$

It is more convenient to define the function $\bar{\psi}$ on the whole ideal $I$. In the following lemma, the domain of $\bar{\psi}$ is extended to $I$. First note that the assumption $X$ and $I$ disjoint implies that every element $f$ of $I$ admits a factorization $f = hgk$ with $h, k \in X^*$ and $g \in V$.

LEMMA 3.7. *For every $f$ in $I$, the element $[\bar{\lambda}^h(g\bar{\psi})\bar{\rho}^k]\xi$ does not depend upon the factorization $f = hgk$ such that $h, k \in X^*$ and $g \in V$. Let $\bar{\psi}$ be the function from $I$ into $Y^+$ which associates $[\bar{\lambda}^h(g\bar{\psi})\bar{\rho}^k]\xi$ with $f = ghk$. Then $\bar{\psi}$ is equal to the restriction of $\eta$ to $I$.*

*Proof.* Let $f$ be in $I$, $h, k$ be in $X^*$ and $g$ be in $V$ such that $f = hgk$.

$$\begin{aligned}[][\bar{\lambda}^h(g\bar{\psi})\bar{\rho}^k]\zeta &= \lambda^{h\gamma}(g\bar{\psi}\gamma)\rho^{k\gamma} && \text{by (L}') \text{ and (R}')\\ &= \lambda^{h\gamma}(g\gamma)\rho^{k\gamma} && \text{since } g\bar{\psi} = g\eta \text{ and } \eta\gamma = \gamma \\ &= h\gamma * g\gamma * k\gamma && \text{by Proposition 3.2} \\ &= (hgk)\gamma \\ &= f\gamma.\end{aligned}$$

Hence,

$$[\bar{\lambda}^h(g\bar{\psi})\bar{\rho}^k]\xi = f\gamma\zeta^{-1}\xi = f\eta. \quad \blacksquare$$

The following proposition sums up Lemmas 3.5 and 3.7.

PROPOSITION 3.4. *Let $U = \langle M, N, \theta, \psi \rangle$ an ideal extension of $N$ by $M$, $\eta$ be the description of $U$ defined in Proposition 3.3, $\bar{\theta} = (\bar{\lambda}, \bar{\rho})$ the mapping associated to $\theta = (\lambda, \rho)$, and $\bar{\psi}$ the function associated to $\psi$. Then,*

$$f\eta = \begin{cases} f\beta & \text{if } f \in X^+ \backslash I \\ f\bar{\psi} & \text{if } f \in I \\ (\bar{\lambda}^g y \bar{\rho}^h)\xi & \text{if } f = gyh,\ g, h \in (X \cup Y)^*,\ y \in Y. \end{cases}$$

### 3.3. *Regular Ideal Extensions*

We keep the notations of the previous section. Suppose that $M$ and $N$ are rational semigroups and that $\beta$ and $\xi$ are rational descriptions of $M$ and $N$.

Let $U=\langle M, N, \theta, \psi\rangle$ an ideal extension of $N$ by $M$ and let $\eta$ be the description defined in the previous section. We first prove that the restriction of $\eta$ to $(X\cup Y)^*Y(X\cup Y)^*$ is a rational function. Hence, whether an ideal extension $\langle M, N, \theta, \psi\rangle$ is rational or not depends only on the rationality of the function $\bar{\psi}$ defined in Lemma 3.7.

We first characterize the left-translations of a free-semigroup.

LEMMA 3.8. *Let $Z$ be a finite set. The monoid $\Lambda(Z^+)$ of left-translations (resp. the monoid $P(Z^+)$ of right-translations) of $Z^+$ is isomorphic to the monoid of column-monomial (resp. row-monomial) matrices of dimension $|Z|$, the entries of which are in $Z^*\cup\{0\}$.*

*Proof.* Let $\lambda$ be a left-translation of $Z^+$. Recall that $\lambda$ is completely determined by its values on the letters of $Z$. For each $t$ in $Z$, the image $\lambda t$ is a non empty word, the first letter of which is a letter $z$. Thus, the matrix $\underline{\lambda}$ defined by

$$\forall s, t\in Z, \qquad \underline{\lambda}_{st}=\begin{cases} w & \text{if } \lambda t=sw \\ 0 & \text{otherwise}\end{cases}$$

is a column-monomial matrix which characterizes $\lambda$. Conversely, any column-monomial matrix defines a left-translation of $Z^+$. Thus we obtain a bijection from $\Lambda(Z^+)$ onto the monoid of column-monomial matrices of dimension $|Z|$, with entries in $Z^*\cup\{0\}$.

Let us now verify that this bijection is a morphism; i.e., $(\underline{\lambda}\underline{\lambda}')_{st}=(\underline{\lambda\lambda'})_{st}$ for every left-translation $\lambda$ and $\lambda'$ and every letter $s$ and $t$:

$$\lambda\lambda' t=\lambda(z'u')=(\lambda z')u'=zuu'.$$

Thus,

$$\begin{aligned}
\underline{\lambda}'_{z't}&=u'\\
\underline{\lambda}_{zz'}&=u\\
(\underline{\lambda\lambda'})_{zt}&=uu'\\
(\underline{\lambda}\underline{\lambda}')_{st}&=\sum_{s'}\underline{\lambda}_{ss'}\underline{\lambda}'_{s't}=\underline{\lambda}_{sz'}u'\\
(\underline{\lambda}\underline{\lambda}')_{st}&=\begin{cases}\underline{\lambda}_{zz'}u'=uu'=(\underline{\lambda\lambda'})_{st} & \text{if } \quad s=z\\ 0=(\underline{\lambda\lambda'})_{st} & \text{otherwise.}\end{cases}
\end{aligned}$$

The isomorphism from $P(Z^+)$ onto the monoid of row-monomial matrices of dimension $|Z|$, with entries in $Z^* \cup \{0\}$, is built in the same way. If $\rho$ is a right-translation of $Z^+$, the matrix $\underline{\rho}$ is defined by

$$\forall s, t \in Z \qquad \underline{\rho}_{ts} = \begin{cases} w & \text{if} \quad t\rho = ws \\ 0 & \text{otherwise} \end{cases}$$

and this gives an isomorphism. ∎

LEMMA 3.9. *Let $U = \langle M, N, \theta, \psi \rangle$ be an ideal extension of a rational semigroup $N$ by a rational semigroup $M$. Let $\beta$ (resp. $\xi$) be a rational description of $M$ (resp. $N$) for a generating system $(X, \alpha)$ (resp. $(Y, \xi)$). Let $\eta$ be the description of $U$ defined in Proposition 3.3. Then the restriction of $\eta$ to $(X \cup Y)^* Y (X \cup Y)^*$ is rational.*

*Proof.* The set $(X \cup Y)^* Y (X \cup Y)^*$ has the rational unambiguous expression $(X \cup Y)^* Y X^*$. Since, by Proposition 3.4, $(gyh)\eta = (\bar{\lambda}^g y \bar{\rho}^h)\xi$ for any $g$ in $(X \cup Y)^*$, $y$ in $Y$, $h$ in $X^*$, the restriction of $\eta$ to $(X \cup Y)^* Y X^*$ is equal to the following composition of functions:

$$\begin{array}{ccccccc} (X \cup Y)^* Y X^* & \to & Y^+ X^* & \to & Y^+ & \to & Y^+ \\ gyh & \mapsto & (\bar{\lambda}^g y) h & \mapsto & (\bar{\lambda}^g y)\bar{\rho}^h & \mapsto & (\bar{\lambda}^g y \bar{\rho}^h)\xi. \end{array}$$

We have thus only to prove the rationality of the functions:

$$\tau \colon (X \cup Y)^* Y \to Y^+$$
$$gy \mapsto \bar{\lambda}^g y$$

and

$$\tau' \colon Y^+ X^* \to Y^+$$
$$vh \mapsto v\bar{\rho}^h.$$

Let $\mu$ be the morphism from $(X \cup Y)^*$ into the monoid of matrices of dimension $|Y|$, the entries of which are in $Y^* \cup \{0\}$, defined by

$$\forall z \in X \cup Y, \qquad z\mu = \underline{\bar{\lambda}}^z$$

Then, by Lemma 3.8, $g\mu = \underline{\bar{\lambda}}^g$ for every $g$ in $(X \cup Y)^*$.

Let $\pi$ be the row-matrix and $\nu$ be the column-matrix of dimension $|Y|$ such that $\pi_y = y$ and $\nu_y = 1$ for each $y$ in $Y$.

Let $g$ be in $(X \cup Y)^*$ and $y$ in $Y$:

$$\pi \cdot (gy)\mu \cdot \nu = \sum_{y', y'' \in Y} y' (\underline{\bar{\lambda}}^{gy})_{y'y''}.$$

Recall that

$$(\underline{\bar{\lambda}}^{gy})_{y'y''} = \begin{cases} w & \text{if } \bar{\lambda}^{gy} y'' = y'w \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\bar{\lambda}^{gy} y'' = \sum_{y' \in Y} y' (\underline{\bar{\lambda}}^{gy})_{y'y''}.$$

Hence,

$$\begin{aligned} \pi \cdot (gy)\mu \cdot v &= \sum_{y'' \in Y} \bar{\lambda}^{gy} y'' \\ &= \sum_{y'' \in Y} (\bar{\lambda}^{g} y)\, y'' \\ &= (gy)\tau \cdot Y. \end{aligned}$$

The relation which associates $\pi \cdot (w\mu) \cdot v$ with each $w$ in $(X \cup Y)^*$ is rational and the function which erases the last letter of a word is rational, hence $\tau$ is rational. We prove the rationality of $\tau'$ in a similar manner. Thus, the restriction of $\eta$ to $(X \cup Y)^* Y X^*$ is rational. ▌

DEFINITION 3.7. Let $U = \langle M, N, \theta, \psi \rangle$, an ideal extension of a rational semigroup $N$ by a rational semigroup $M$, and let $\bar{\psi}$ be the function associated with the ramification $\psi$. $U$ is called a *regular ideal extension* of $N$ by $M$ if $\bar{\psi}$ is a rational function.

The reader may wonder why we use the word "regular" instead of "rational" in this definition. It is not because we suddenly abandon our terminology but simply because, otherwise, Theorem 3.1 below will sound like a mere tautology: "an ideal extension is rational if it is a rational ideal extension."

EXAMPLE 3.2. A non-regular ideal extension of a rational semigroup by a rational semigroup.

Let $M$ be the Rees quotient of $X^* = \{a, b\}^*$ by the rational ideal $I = X^*ab^*aX^*$. By Proposition I.6.1, $M$ is a rational monoid.

$$M = b^* \cup b^*ab^* \cup \{0_M\}.$$

Let $N$ be the two elements semigroup:

$$N = \{c, 0\}, \qquad c^2 = 0.$$

$N$ is the Rees quotient of $Y^+ = \{c\}^+$ by the ideal $c^2c^*$ and $N$ is a rational semigroup. Let us now construct an ideal extension of $N$ by $M$ which is not regular.

Let $\theta$ be the representation of $M^{\cdot}$ by permutable bitranslations of $N$, which maps any element of $M$ different of $1_M$ onto the unique inner bitranslation $\pi_c = (\lambda_c, \rho_c)$ of $N$ (and $1_M$ onto $(\mathrm{id}_N, \mathrm{id}_N)$). Let $\psi$ be the function from $W_M = b^*ab^* \times b^*ab^*$ into $N$ defined by

$$(b^k ab^n, b^m ab^l)\psi = 0 \qquad \text{if} \quad k \neq 0 \text{ or } l \neq 0$$

$$(ab^n, b^m a)\psi = \begin{cases} c & \text{if} \quad n+m \text{ is a prime} \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\psi$ is a ramification adapted to $\theta$.

Let $\bar{\psi}$ be the function from $I = X^*ab^*aX^*$ into $Y^+\xi = \{c, c^2\}$, associated with $\psi$. Then,

$$(ab^n a)\bar{\psi} = \begin{cases} c & \text{if } n \text{ is a prime} \\ c^2 & \text{otherwise.} \end{cases}$$

Hence $\bar{\psi}$ is not rational and $U = \langle M, N, \theta, \psi \rangle$ is not a regular ideal extension of $N$ by $M$.

Example 3.3. A regular ideal extension of a rational semigroup by a rational semigroup.

$M, X, I, N, Y, \xi, \theta$ are the same ones as in the previous example.

Let $\psi$ be the function from $W_M$ into $N$ defined by

$$(b^k ab^n, b^m ab^l)\psi = 0 \qquad \text{if} \quad k \neq 0 \text{ or } l \neq 0$$

$$(ab^n, b^m a)\psi = \begin{cases} c & \text{if} \quad n+m \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\psi$ is a ramification adapted to $\theta$. Let $\bar{\psi}$ be the function associated with $\psi$. Then,

$$(uab^n av)\bar{\psi} = c^2 \qquad \text{if} \quad u \neq 1 \text{ or } v \neq 1$$

$$(ab^n a)\bar{\psi} = \begin{cases} c & \text{if } n \text{ is even} \\ c^2 & \text{otherwise.} \end{cases}$$

The function $\bar{\psi}$ is rational and $U = \langle M, N, \theta, \psi \rangle$ is a regular ideal extension of $N$ by $M$.

Theorem 3.1. *An ideal extension of a rational semigroup by a rational semigroup is a rational semigroup if and only if it is a regular ideal extension.*

*Proof.* Let $M$ and $N$ be two rational semigroups and let $\beta$ and $\xi$ be rational descriptions of $M$ and $N$.

Let $U = \langle M, N, \theta, \psi \rangle$ an ideal extension of $N$ by $M$.

*The condition is sufficient.* Suppose $U$ is a regular ideal extension. The description $\eta$ is given by

$$f\eta = \begin{cases} f\beta & \text{if } f \in X^+ \setminus I \\ f\bar{\psi} & \text{if } f \in I \\ (\bar{\lambda}^g y \bar{\rho}^h)\xi & \text{if } f = gyh \text{ with } g, h \in (X \cup Y)^* \text{ and } y \in Y. \end{cases}$$

By property (P4), $I = 0_M \alpha^{-1}$ is a rational set of $X^*$. Since $\beta$ is rational, its restriction to the rational set $X^+ \setminus I$ is rational. By hypothesis, $\bar{\psi}$ is rational. By Lemma 3.9, the restriction of $\eta$ to $(X \cup Y)^* Y (X \cup Y)^*$ is rational. Thus, $\eta$ is a rational description of $U$.

*The condition is necessary.* Suppose $U$ is a rational semigroup. The function $\bar{\psi}$ is the restriction to $I$ of the function $\gamma\zeta^{-1}\xi$ and $I$ is rational (property (P4)). Hence, we have only to prove that $\gamma\zeta^{-1}\xi$ is rational. Since $\zeta^{-1}$ is a rational function from $U$ into $(X \cup Y)^+$ and $\xi$ is a rational function from $(X \cup Y)^+$ into $(X \cup Y)^+$, $\zeta^{-1}\xi$ is a rational function from $U$ into $(X \cup Y)^+$. $\gamma$ is a rational function from $(X \cup Y)^+$ into $U$. Since $U$ is a rational semigroup, the product $\gamma\zeta^{-1}\xi$ is a rational function from $(X \cup Y)^+$ into $(X \cup Y)^+$ (by property (P5)). ∎

Example 3.2 (and Theorem 3.1) shows that there exist ideal extensions of a rational semigroup by a rational semigroup which are not rational.

*Remark* 3.5. Since $\bar{\psi}$ is expressed in terms of the mappings $\alpha$, $\beta$, $\zeta$, and $\xi$, it may appear that the property for an ideal extension to be regular depends upon these elements. It is a consequence of Theorem 3.1 that this property is indeed intrinsic to the ideal extension $U$ and does not depend upon the descriptions chosen for $M$ and $N$.

The following corollary makes the whole theory coherent:

COROLLARY 3.1. *Let $M$ be a rational monoid and $I$ be an ideal of $M$, finitely generated as a subsemigroup of $M$. Then $M$ is a regular ideal extension of $I$ by $M/\!/I$.*

*Proof.* Since $I$ is a finitely generated subsemigroup of $M$ and since $M$ is a rational monoid, $I$ and $M/\!/I$ are rational semigroups. $M$ is thus an ideal extension of rational semigroups which is a rational monoid. By Theorem 3.1, $M$ is a regular ideal extension. ∎

The following corollaries deal with some particular cases of ideal extension, in which $\bar{\psi}$ happens to be rational.

COROLLARY 3.2. *Any ideal extension of a rational semigroup by a rational semigroup without zero is a rational semigroup.*

*Proof.* The domain of the function $\bar{\psi}$ is empty and $\bar{\psi}$ is thus rational. ▮

COROLLARY 3.3. *Any ideal extension of a rational* monoid *by a rational semigroup is a rational semigroup.*

*Proof.* Let $\langle M, N, \theta, \psi \rangle$ be an ideal extension of a monoid $N$ by a semigroup $M$, and $u = 1_N \zeta^{-1} \xi$. We prove that

$$f\bar{\psi} = (\bar{\lambda}^f u)\xi \qquad \text{for each } f \text{ in } I.$$

Let $f$ be in $I$, then $f\gamma \in N$,

$$\begin{aligned} (\bar{\lambda}^f u)\zeta &= \lambda^{f\alpha}(u\zeta) && \text{by (L')} \\ &= (f\gamma)\, 1_N, && \text{since } f\gamma \in N \\ &= f\gamma. \end{aligned}$$

Hence, $f\bar{\psi} = f\gamma\zeta^{-1}\xi = (\bar{\lambda} fu)\xi$ and $\bar{\psi}$ is equal to the following product of rational relations:

$$\begin{array}{ccccccc} I & \to & X^+ Y^+ & \to & Y^+ & \to & Y^+ \\ f & \mapsto & fu & \mapsto & \bar{\lambda}^f u & \mapsto & (\bar{\lambda}^f u)\xi. \end{array}$$

Thus, the function $\bar{\psi}$ is rational and $\langle M, N, \theta, \psi \rangle$ is a regular ideal extension of $N$ by $M$.

LEMMA 3.10. *Let $G$ be a rational set of generators of the ideal $I = 0_M \alpha^{-1}$. An ideal extension $\langle M, N, \theta, \psi \rangle$ is regular if an only if the restriction $\bar{\psi}|_G$ of $\bar{\psi}$ to $G$ is rational.*

*Proof. The condition is sufficient.* Let $f$ be in $I$, then $f$ has a factorization $f = hgk$ with $h, k$ in $X^*$ and $g$ in $G$.

$$\begin{aligned} [\bar{\lambda}^h(g\bar{\psi})\,\bar{\rho}^k]\zeta &= \lambda^{h\gamma}(g\bar{\psi}\gamma)\,\rho^{k\gamma} && \text{by (L') and (R')} \\ &= \lambda^{h\gamma}(g\gamma)\,\rho^{k\gamma} \\ &= h\gamma * g\gamma * k\gamma \\ &= (hgk)\gamma \\ &= f\gamma. \end{aligned}$$

Thus $f\bar{\psi} = f\gamma\zeta^{-1}\xi = [\bar{\lambda}^h(g\bar{\psi})\,\bar{\rho}^k]\xi$ and $\bar{\psi}$ is equal to the following product of rational relations:

$$\begin{array}{ccccccc} I = X^*GX^* & \to & X^*Y^+X^* & \to & Y^+ & \to & Y^+ \\ f = \quad hgk & \mapsto & h(g\bar{\psi})k & \mapsto & [\bar{\lambda}^h(g\bar{\psi})]\bar{\rho}^k & \mapsto & [\bar{\lambda}^h(g\bar{\psi})\,\bar{\rho}^k]\xi. \end{array}$$

Thus, the function $\bar{\psi}$ is rational.

*The condition is necessary.* Since $\bar{\psi}$ is rational and $G$ is rational, $\bar{\psi}|_G$ is rational. ▌

COROLLARY 3.4. *With the current notations, if $I$ is a finitely generated ideal of $X^*$, any ideal extension of $N$ by $M$ is a rational semigroup.*

COROLLARY 3.5. *Any ideal extension of a rational semigroup by a finite semigroup is a rational semigroup.*

*Proof.* Let $M$ be a finite semigroup and $U = \langle M, N, \theta, \psi \rangle$ an ideal extension of $N$ by $M$. The set $G = I \backslash (X^*IX^+ \cup X^+IX^*)$ is a rational set of generators of $I$. By Lemma 3.10, we have just to prove that the restriction of $\eta$ to $G$ is rational. Let $\tau$ be the relation from $(X \cup Y)^*$ into $M \times M$ defined by

$$f\tau = \{(f_1\alpha, f_2\alpha) \mid f_1, f_2 \in X^+ \text{ and } f = f_1 f_2\}$$

Then $\bar{\psi}|_G$ is equal to the following product of functions:

$$(X \cup Y)^+ \xrightarrow{\tau|_G} M \times M \xrightarrow{\psi} N \xrightarrow{\zeta^{-1}} (X \cup Y)^+ \xrightarrow{\xi} (X \cup Y)^+.$$

All the semigroups which appear in this composition are rational, hence, by property (P5), we just have to prove the rationality of the relations.

$\tau|_G$ is rational since $G$ is rational and the graph of $\tau$ is

$$\hat{\tau} = \bigcup_{(m_1, m_2) \in M \times M} \{(m_1\alpha^{-1})(m_2\alpha^{-1}) \times (m_1, m_2)\}.$$

$\psi$ is rational since it is a function defined in a finite semigroup. $\zeta^{-1}$ is rational since $\zeta$ is a morphism and $\xi$ is rational by hypothesis. ▌

COROLLARY 3.6. *Any ideal extension of a free semigroup by a rational semigroup is a rational semigroup.*

*Proof.* Let $N = Y^+$ and $\zeta = \xi = \mathrm{id}_{Y^+}$. Let $V$ defined as in Lemma 3.6; i.e.,

$$V = (X^+ \backslash I)(X^+ \backslash I) \cap I.$$

Let $g = g_1 g_2$ in $V$ (with $g_1, g_2$ in $X^+ \backslash I$). Let $y$ be in $Y$:

$$\begin{aligned}
g\bar{\psi} &= (g_1\alpha, g_2\alpha)\,\psi\zeta^{-1}\xi && \text{by definition of } \bar{\psi} \text{ on } V \\
&= (g_1\alpha, g_2\alpha)\psi && \text{since } \zeta = \xi = \mathrm{id}_{Y^+} \\
&= (\lambda^{g_1\alpha}\lambda^{g_2\alpha}y)\,y^{-1} && \text{by Remark 3.3} \\
&= (\lambda^{g_1\lambda}\lambda^{g_2\alpha}(y\zeta))\,y^{-1} && \text{since } \zeta = \mathrm{id}_{Y^+} \\
&= (\bar{\lambda}^g y)\,y^{-1} && \text{by (L).}
\end{aligned}$$

Hence the restriction of $\bar{\psi}$ to $V$ is rational. Since $V$ is a rational set of generators of $I$, the extension is regular (by Lemma 3.10). ∎

### 3.4. *An Application of the Construction of Ideal Extension*

In Part I, it has been proved that any rational monoid $M$ has the property that Rat $M =$ Rec $M$, that is to say that Kleene's theorem holds in every rational monoid, as it is the case in finitely generated free monoids. Moreover, it was noted that the few examples of monoids known so far and in which Kleene's theorem holds belong to the family of rational monoids (Examples I.4.1 and I.6.1). However, this is not the general case. The ideal extensions give a process to build monoids $M$ such that Rat $M =$ Rec $M$ and that are not rational. This is the answer to a problem stated in Part I.

More precisely, Theorem 3.2 below characterizes among ideal extensions of rational semigroups those in which Kleene's theorem holds. This is done in terms of the ramification function. Example 3.4 gives such a ramification function which is not rational and thus answers the question by Theorem 3.1.

LEMMA 3.11. *Let $S$ and $U$ be two semigroups, $\gamma$ a surjective morphism from $S$ onto $U$, and $R$ a subset of $U$. If $R\gamma^{-1} \in$ Rec $S$, then $R \in$ Rec $U$.*

*Proof.* The lemma is a consequence of the fact that $R$ and $R\gamma^{-1}$ have the same syntactic semigroup (cf. Lemma I.4.1). ∎

THEOREM 3.2. *Let $M$ and $N$ be two rational semigroups. Let $\beta$ be a rational description of $M$ for a generating system $(X, \alpha)$ and $\xi$ be a rational description of $N$ for a generating system $(Y, \zeta)$. Let $U = \langle M, N, \theta, \psi \rangle$ be an ideal extension of $N$ by $M$ and let $\bar{\psi}$ be the function associated to the ramification $\psi$. $U$ is a semigroup in which Kleene's theorem holds if and only if $\bar{\psi}$ and $\bar{\psi}^{-1}$ map rational sets into rational sets.*

*Proof.* *The condition is sufficient.* Assume that $\bar{\psi}$ and $\bar{\psi}^{-1}$ map rational sets into rational sets.

Since $U$ is finitely generated, $\operatorname{Rec} U \subset \operatorname{Rat} U$. In order to prove the reverse inclusion, let $R$ be in $\operatorname{Rat} U$. Then, there exists $K \in \operatorname{Rat}(X \cup Y)^+$ such that $R = K\gamma$. By the Lemma 3.11, it is sufficient to prove that $R\gamma^{-1} = K\gamma\gamma^{-1} = K\eta\eta^{-1}$ is in $\operatorname{Rat}(X \cup Y)^+$. Let us recall that

$$f\eta = \begin{cases} f\beta & \text{if } f \in X^+ \setminus I \\ f\bar{\psi} & \text{if } f \in I \\ (\bar{\lambda}^g y \bar{\rho}^h)\xi & \text{if } f = gyh \text{ with } g, h \in (X \cup Y)^* \text{ and } y \in Y. \end{cases}$$

The restriction of $\eta$ to $I$ is equal to $\bar{\psi}$. Let $\eta_1$ be the restriction of $\eta$ to $(X \cup Y)^+ \setminus I$. Since $\eta_1$ is rational and $\bar{\psi}$ preserves rationality, $\eta$ preserves rationality. Since $\eta_1^{-1}$ is rational and $\bar{\psi}^{-1}$ preserves rationality, $\eta^{-1}$ preserves rationality. Thus $K\eta\eta^{-1} \in \operatorname{Rat}(X \cup Y)^+$.

*The condition is necessary.* Assume that Kleene's theorem holds in $U$.

Let $K \in \operatorname{Rat}(X \cup Y)^+$,

$$K\bar{\psi} = (K \cap I)\eta = (K \cap I)\gamma\zeta^{-1}\xi,$$

$K \cap I \in \operatorname{Rat}(X \cup Y)^+$; hence $(K \cap I)\gamma \in \operatorname{Rat} U$. Since $\operatorname{Rat} U = \operatorname{Rec} U$, $(K \cap I)\gamma \in \operatorname{Rec} U$, hence $(K \cap I)\gamma\zeta^{-1} \in \operatorname{Rat}(X \cup Y)^+$. Since $\xi$ is rational, by hypothesis, $(K \cap I)\gamma\zeta^{-1}\xi \in \operatorname{Rat}(X \cup Y)^+$. Hence $K\bar{\psi}$ is rational,

$$K\bar{\psi}^{-1} = K\eta^{-1} \cap I = K\gamma\gamma^{-1} \cap I.$$

Since $\operatorname{Rat} U = \operatorname{Rec} U$, $K\gamma \in \operatorname{Rec} U$. Hence $K\gamma\gamma^{-1} \in \operatorname{Rat}(X \cup Y)^+$ and $K\bar{\psi}^{-1}$ is rational. ∎

EXAMPLE 3.4. Let $M$ be the Rees quotient of $X^* = \{x, y, z\}^*$ by the rational ideal $I = X^*z\{x, y\}^*zX^*$, $M$ is a rational monoid (cf. Proposition I.6.1). We identify $M^{\bullet}$ and its trace in $X^*$; then

$$M = \{x, y\}^* \cup \{x, y\}^*z\{x, y\}^* \cup \{0_M\}.$$

Let $Y = \{a, b, c\}$ be a copy of $X$. Any ideal extension of $Y^+$ by $M$ is a rational semigroup (Corollary 3.6); that is why we consider a Rees quotient of $Y^+$. Let $N$ be the Rees quotient of $Y^+$ by the rational ideal $J = Y^*c\{a, b\}^*cY^+ \cup Y^+c\{a, b\}^*cY^*$. Thus $N$ is a rational semigroup. We identify $N^{\bullet}$ and its trace in $Y^*$. Then

$$N = \{a, b\}^+ \cup \{a, b\}^*c\{a, b\}^* \cup c\{a, b\}^*c \cup \{0_N\}.$$

*Definition of $\theta$.* Let $\varepsilon$ be the mapping from $M \setminus \{0_M, 1_M\}$ into $N$, which writes the letters $a, b, c$ instead of the letters $x, y, z$. With every $m$ in $M \setminus \{0_M, 1_M\}$, we associate the inner bitranslation $(\lambda_{m\varepsilon}, \rho_{m\varepsilon})$ of $N$ and we associate $(\mathrm{id}_N, \mathrm{id}_N)$ with $1_M$. Thus we obtain a representation $\theta$ of $M^{\cdot}$ by permutable bitranslations of $N$.

*Definition of* $\psi$. We want to build a function $\psi$ such that $\bar{\psi}$ is not rational, but $\bar{\psi}$ and $\bar{\psi}^{-1}$ map rational sets into rational sets. An example of a non-rational function which preserves the rational sets is the function reversal. For each element $m$ of $M^{\cdot}$, let $\tilde{m}$ be its reversal (in fact, the reversal of its representative in $\{x, y\}^* \cup \{x, y\}^*z\{x, y\}^*$),

$$W_M = \{x, y\}^*z\{x, y\}^* \times \{x, y\}^*z\{x, y\}^*.$$

The definition of $M$ and $N$ allows us to take, for $\psi$, any function built in the following way:

— for an element $f$ of $H = z\{x, y\}^* \times \{x, y\}^*z$, $f\psi$ is any element of $c\{a, b\}^*c$

— for an element $f$ of $W_M \backslash H$, $f\psi$ has the value $0_N$ (this condition is implied by (C1), (C2), and (C3)).

Let $\psi$ be the mapping from $W_M$ into $N$ defined by:

$$\begin{array}{lll} (zu, vz)\psi = (\widetilde{zuvz})\varepsilon & \text{for} & u, v \text{ in } \{x, y\}^* \\ (u'zu, vzb')\psi = 0_N & \text{for} & u, u', v, v' \text{ in } \{x, y\}^* \\ & & \text{such that } u' \neq 1_M \text{ or } v' \neq 1_M. \end{array}$$

By construction, $\psi$ is a ramification adapted to $\theta$. Thus $U = \langle M, N, \theta, \psi \rangle$ is an ideal extension of $N$ by $M$.

Let us now prove that $U$ is a non-rational monoid, in which Kleene's theorem holds. The mapping $\xi$ from $Y^+$ into $Y^+$ defined by

$$u\xi = \begin{cases} u & \text{if } u \in \{a, b\}^+ \cup \{a, b\}^*c\{a, b\}^* \cup c\{a, b\}^*c \\ c^3 & \text{otherwise} \end{cases}$$

is a rational description of $N$.

Let $\bar{\psi}$ be the function from $I = X^*z\{x, y\}^*zX^*$ into $Y^+\xi = \{a, b\}^+ \cup \{a, b\}^*c\{a, b\}^* \cup c\{a, b\}^*c \cup \{c^3\}$ associated with $\psi$. Then

$$f\bar{\psi} = \begin{cases} \tilde{f}\varepsilon & \text{if } f \in z\{x, y\}^*z \\ c^3 & \text{if } f \in I \backslash z\{x, y\}^*z. \end{cases}$$

It is clear that $\bar{\psi}$ is a non-rational function and that $\bar{\psi}$ and $\bar{\psi}^{-1}$ map rational sets into rational sets; therefore, $U$ is not a rational monoid but Kleene's theorem holds in $U$.

## 4. Rational Thin Monoids

Rational monoids, as we said, have an "easy" multiplication. Here are studied rational monoids that have in addition a very "simple" trace. Redei and ideal extensions are appropriate constructions to obtain such monoids.

### 4.1. *Thin Monoids*

Let $M$ be a monoid. A *ray* of $M$ is either a singleton or a subset $uv^*w$ (where $u$, $v$, $w$ are in $M$). A *proper ray* of $M$ is either a singleton or a ray $uv^*w$ such that

$$uv^n w = uv^m w \Rightarrow n = m.$$

A subset of $M$ is *thin* if it is a finite union of disjoint proper rays of $M$. Note that a thin subset of a monoid $M$ is a rational subset of $M$. Note also that the image of a thin subset by a morphism is thin if the restriction of the morphism to this subset is one-to-one.

Thin subsets of a monoid $M$ are characterized by their traces in any generating system of $M$ by the following Lemma (cf. Sakarovitch, 1979):

LEMMA 4.1. *Let $M$ be a monoid and $(X, \alpha)$ be a generating system of $M$. A subset $R$ of $M$ is thin if and only if there exists a thin trace of $R$ for $(X, \alpha)$.*

*Proof.* *The condition is sufficient.* Assume that $R$ has a thin trace $T$ for a generating system $(X, \alpha)$. The restriction of the morphism $\alpha$ to the thin subset $T$ is injective, hence the image of $T$, i.e., $R$, is thin.

*The condition is necessary.* Assume that $R$ is a thin subset of $M$. $R$ can be written as a finite union of disjoint proper rays:

$$R = \left(\bigcup_{i \in I} u_i v_i^* w_i\right) \cup \left(\bigcup_{j \in J} t_j\right),$$

where $u_i$, $v_i$, $w_i$, and $t_j$ are elements of $M$. The set $\{u_i, v_i, w_i, t_j \mid i \in I, j \in J\}$ can be completed in a finite set of generators of $M: G = \{u_i, v_i, w_i, t_j, s_k \mid i \in I, \ j \in J, k \in K\}$. Let $A = \{a_i, b_i, c_i, d_j, e_k \mid i \in I, \ j \in J, k \in K\}$ be an alphabet in an one-to-one correspondance with $G$ and let $\zeta$ be the morphism from $A^*$ onto $M$ defined by: $a_i\zeta = u_i$, $b_i\zeta = v_i$, $c_i\zeta = w_i$, $d_j\zeta = t_j$, and $e_k\zeta = s_k$. Then $T = (\bigcup_{i \in I} a_i b_i^* c_i) \cup (\bigcup_{j \in J} d_j)$ is a thin trace of $R$ for $(A, \zeta)$.

Let $(X, \alpha)$ be any generating system of $M$. There exists a morphism $\phi$ from $A^*$ into $X^*$ such that $\phi\alpha = \zeta$. Since $\zeta$ is injective on $T$, $\phi$ is injective on $T$. Thus $T\phi$ is a thin subset of $X^*$. Moreover, $T\phi$ is a trace of $R$ for $(X, \alpha)$. ▌

The following properties relate thin subsets of the free monoids, rational relations, and rational sets.

PROPOSITION 4.1 (Courcelle, 1974). *Let $R$ and $S$ be two subsets of free monoids in rational bijection. Then, if one is thin so is the other. Any rational subset of a free monoid included in a thin subset is thin.*

From Proposition 4.1 follows that rational monoids and free monoids have the same properties concerning thin subsets.

LEMMA 4.2. *In a rational monoid, a rational subset included in a thin subset is thin.*

*Proof.* Let $M$ be a rational monoid and $K$ a rational subset of $M$ included in a thin subset $R$. Let $(X, \alpha)$ be a generating system of $M$. Then, by Lemma 4.1, there exists a thin trace $T$ of $R$ for $(X, \alpha)$. Since $K$ is a rational set of a rational monoid, $K\alpha^{-1}$ is a rational set $X^*$. $T' = K\alpha^{-1} \cap T$ is a rational set of a free monoid included in a thin set, hence $T'$ is thin. Moreover, $T'$ is a trace of $K$ for $(X, \alpha)$. $K$ is thus a thin subset of $M$. ∎

LEMMA 4.3. *In a rational monoid, any finite union of rays is thin.*

*Proof.* Let $M$ be a rational monoid and $R = (\bigcup_{i \in I} u_i v_i^* w_i) \cup (\bigcup_{j \in J} t_j)$ a finite union of rays of $M$. Let $X = \{a_i, b_i, c_i, d_j \mid i \in I, j \in J\}$ be an alphabet and let $\alpha$ be the morphism from $X^*$ into $M$ defined by: $a_i\alpha = u_i$, $b_i\alpha = v_i$, $c_i\alpha = w_i$, and $d_j\alpha = t_j$. Since a finitely generated submonoid of a rational monoid is a rational monoid we may assume that $(X, \alpha)$ is a generating system of $M$. Let $T = (\bigcup_{i \in I} a_i b_i^* c_i) \cup (\bigcup_{j \in J} d_j)$. Since $M$ is a rational monoid, there exists a rational subset $K$ of $X^*$ such that $K \subset T$ and $\alpha$ is a bijection from $K$ onto $T\alpha = R$. By Proposition 4.1, $K$ is thin. Since the restriction of $\alpha$ to $K$ is a bijection from $K$ onto $R$, $R$ is thin. ∎

LEMMA 4.4. *Let $M$ be a rational monoid and $(X, \alpha)$ any generating system of $M$. Any rational trace of a thin subset of $M$ is thin.*

*Proof.* Let $R$ be a thin subset of $M$. Since $R$ is thin, $R$ has a thin trace for $(X, \alpha)$. Since $M$ is a rational monoid, all the rational traces of $R$ are in rational bijection (it is a consequence of property (P5)). One of them is thin hence all are thin. ∎

DEFINITION 4.1. A monoid $M$ is thin if it is a thin subset of itself.

This definition implies that a thin monoid is finitely generated and Lemma 4.1 implies that a monoid $M$ is a thin monoid iff it has thin traces for any generating system $(X, \alpha)$.

EXAMPLE 4.1. Thin monoids:

• the free monoid $a^*$, isomorphic to $\mathbf{N}$

• the free group with one generator $\mathbf{Z}$. As a monoid $\mathbf{Z}$ is a quotient of $\{a, b\}^*$ and has there the trace $a^* \cup bb^*$.

• the monoids $M_{p,q}$ defined in Example 2.5. It can be verified that $M_{p,q}$ is isomorphic to the quotient of $\mathbf{N}^2$, the free commutative monoid with two generators $x$ and $y$, by the relation $x^p = y^q$. The trace of $M_{p,q}$ in $\{x, y\}^*$ is $y^* \cup xy^* \cup \cdots \cup x^{p-1}y^*$.

• the direct product of a thin monoid by a finite monoid.

EXAMPLE 4.2. Non-thin monoids:

• the free monoid on a 2-letter alphabet

• the free commutative monoid on a 2-letter alphabet.

Clearly a *rational thin monoid* is a monoid which is both a rational monoid and a thin monoid. As a corollary of Lemma 4.4 any rational trace of a rational thin monoid is a thin subset.

LEMMA 4.5. *Let $M$ be a submonoid of a monoid $N$. If $M$ is a thin subset of $N$ and if $M$ is a rational monoid, then $M$ is a thin subset of itself* (*and thus a rational thin monoid*).

*Proof.* Let $M$ be a submonoid of $N$ such that $M$ is a thin subset of $N$ and a rational monoid. Let $(X, \alpha)$ be a generating system of $N$. Then, by Lemma 4.1, there exists a thin trace $T$ of $M$ for $(X, \alpha)$. Let $(Y, \zeta)$ be a generating system of $M$ and let $T'$ be a rational trace of $M$ for $(Y, \zeta)$. Since $T$ and $T'$ are rational traces of a rational monoid, they are in rational bijection and, since $T$ is thin, $T'$ is thin in $Y^*$ (Proposition 4.1). $M$ has a thin trace for a generating system of $M$ thus $M$ is a thin subset of itself. Hence $M$ is a thin monoid. ∎

The hypothesis that $M$ is a rational monoid is necessary, as shown by the following example:

EXAMPLE 4.3. The basis of the construction is a surjective mapping $\varphi$ from $\mathbf{N}$ onto $\mathbf{N}$ such that its restriction to any infinite rational subset of $\mathbf{N}$ is not one-to-one. Any (surjective) mapping which is constant on intervals of non bounded length is suited. Take, for instance,

$$\varphi : \mathbf{N} \to \mathbf{N}$$
$$n \in [2^k, 2^{k+1}[ \mapsto k+1$$
$$0 \mapsto 0.$$

Now let $X = \{a, b, x\}$ and $\theta$ the congruence on $X^*$ generated by

$$x^2 = x$$

$$uxv = ux \qquad \text{if} \quad u \in \{a, b\}^+$$

$$x(ab)^n x = xab^{n\varphi}x.$$

Let $N$ be the quotient of $X^*$ by $\theta$. We may identify $N$ and $A^+x \cup xA^* \cup x(A^+ \backslash ab(ab)^+)x \cup A^*$ (where $A = \{a, b\}$). Let $M$ be the submonoid of $N$ generated by $x$ and $ab$. Then

$$M = (ab)^+ \cup x(ab)^* \cup xab^*x \cup (ab)^*.$$

$M$ is a finite union of disjoint proper rays of $N$, hence $M$ is a thin subset of $N$. But $M$ is not thin in $M$, since $x(ab)^*x$ $(= xab^*x)$ is not contained in a finite union of disjoint proper rays of $M$ ($b$ is *not* in $M$). Assume the contrary, then $x(ab)^*x \subset \bigcup_{i \in I} u_i v_i^* w_i$ where $I$ is finite and $u_i, v_i, w_i$ are elements of $M$. Since $I$ is finite and since $x(ab)^*x$ is infinite, there exists an element $i$ of $I$ such that $u_i v_i^* w_i$ is infinite. This implies that $u_i = x(ab)^{n_1}$ and $v_i = (ab)^{n_2}$ (with $n_1 \in \mathbf{N}$, $n_2 \in \mathbf{N}$ $n_2 \neq 0$). Any choice of $w_i$ is equivalent to $w_i = (ab)^{n_3}x$ (with $n_3 \in \mathbf{N}$). Hence: $u_i v_i^* w_i = x(ab)^K x$, where $K$ is the rational set $n_1 + \mathbf{N}n_2 + n_3$. Since the restriction of $\varphi$ to $K$ is not one-to-one, $u_i v_i^* w_i$ is not a proper ray. Hence $M$ is not thin in itself.

COROLLARY 4.1. *Let $N$ be a rational thin monoid and $M$ be a finitely generated submonoid of $N$ then $M$ is a rational thin monoid.*

*Proof.* Since $M$ is finitely generated, $M$ is a rational subset of $N$. $N$ is thin, hence, by Lemma 4.2, $M$ is a thin subset of $N$. Since $N$ is a rational monoid and $M$ is a finitely generated submonoid of $N$, $M$ is a rational monoid. By Lemma 4.4, $M$ is a rational thin monoid. ∎

### 4.2. *Languages Recognized by Rational Thin Monoid*

We refer to Eilenberg (1974) and Harrison (1978) for definitions and results in formal language theory. Let us recall that "(formal) language" is another name for a subset of a free monoid and the following definition: a language $L$ of $X^*$ is recognized by a monoid $M$ if there exists a morphism $\phi$ from $X^*$ into $M$ such that $L = L\phi\phi^{-1}$. The *syntactic monoid* of a language is the "smallest" monoid that recognizes the language. A language is rational if and only if its syntactic monoid is finite.

We call *algebraic language* what Harrison (and some others) call context-free language. As a consequence of Parikh theorem, we have:

LEMMA 4.6. *In a free monoid, the intersection of an algebraic language and a thin language is a thin language (and thus rational).*

From this follows:

PROPOSITION 4.2. *An algebraic language recognized by a rational thin monoid is rational.*

*Proof.* Let $L$ be an algebraic language of $X^*$ and $M$ be a rational thin monoid that recognizes $L$. Let $\alpha$ be a morphism from $X^*$ into $M$ such that $L = L\alpha\alpha^{-1}$. The image of $X^*$ by $\alpha$ is a finitely generated submonoid of the rational thin monoid $M$ and thus is a rational thin monoid (by Corollary 4.1). Hence, we may assume that $\alpha$ is a surjective morphism from $X^*$ onto $M$, i.e., that $(X, \alpha)$ is a generating system of $M$. By Lemma 4.1, there exists a thin trace $T$ of $M$ for $(X, \alpha)$. $T$ is a trace of $M$ and $L = L\alpha\alpha^{-1}$, hence $L\alpha = (L\alpha\alpha^{-1} \cap T)\alpha = (L \cap T)\alpha$. Since $T$ is thin and $L$ is algebraic, $L \cap T$ is rational (by Lemma 4.5). Hence $L\alpha$ is rational. Since $M$ is a rational monoid, $L\alpha\alpha^{-1} = L$ is rational. ∎

COROLLARY 4.2. *An infinite rational thin monoid cannot be the syntactic monoid of an algebraic language.*

EXAMPLE 4.4 (Perrot, 1977). $\mathbf{N}$ and the monoids $M_{p,q}$ cannot be syntactic monoids of algebraic languages.

*Remark* 4.1. An infinite thin monoid or an infinite rational monoid may well be the syntactic monoid of an algebraic language. For instance, the thin monoid $\mathbf{Z}$ is the syntactic monoid of the (full) Dyck language over one letter. The free (and thus rational) monoid $\{a, b\}^*$ is the syntactic monoid of the language $PAL = \{f \in \{a, b\}^* \mid f = \tilde{f}\}$.

## 4.3. *A Family of Rational Thin Semigroups*

The purpose of this paragraph is to describe a family of semigroups that are all rational and thin. These semigroups are obtained from $\mathbf{N}$ by successive ideal and Redei extensions by finite semigroups. More precisely, let $\mathscr{F}$ be the family of semigroups defined in the following way:

$$\mathscr{F}_0 = \{\mathbf{N}_+\}$$

$$\begin{aligned}\mathscr{F}_{n+1} = \{U \mid \exists M \in \mathscr{F}_n, \exists F \text{ finite semigroup such that} \\ U \text{ is a Redei extension of } M \text{ by } F \text{ or} \\ U \text{ is an ideal extension of } M \text{ by } F\}\end{aligned}$$

$$\mathscr{F} = \bigcup_{n \in \mathbf{N}} \mathscr{F}_n.$$

We have:

PROPOSITION 4.3. *The finitely generated semigroups of $\mathscr{F}$ are rational thin semigroups.*

As a remark, we may note that the preceding results on rational and on thin monoids immediately give:

LEMMA 4.7. *An ideal extension of a rational thin semigroup by a finite semigroup is a rational thin semigroup.*

*Proof.* Such an extension is rational (by Corollary 3.5) and thin since it is the union of a thin semigroup and of a finite set. ∎

LEMMA 4.8. *A finitely generated Redei extension of a rational thin semigroup by a finite semigroup is a rational thin semigroup.*

*Proof.* Such a Redei extension is a rational semigroup (Theorem 2.3) and is thin in the associated monoidal Redei extension; hence that is a rational thin semigroup (Lemma 4.5). ∎

Thus, what has to be done to establish Proposition 4.3 is to prove that any finitely generated element of $\mathscr{F}$ can be obtained by a sequence of extensions where every intermediate semigroup is rational and thin.

LEMMA 4.9. *Let $M$ be a semigroup, $F$ be a finite semigroup and $U$ be a Redei extension of $M$ by $F$. If $U$ is finitely generated, then $M$ is finitely generated.*

*Proof.* Let $G$ be a finite set of generators of $U$ and let

$$G_1 = \{m \in M \mid \exists f \in F, (f, m) \in G\}.$$

Let

$$H = G_1 \cup \{s_{f,g} \mid f, g \in F\} \cup \{m^f \mid m \in G_1, f \in F\}.$$

Then $H$ is a finite set of generators of $M$. ∎

LEMMA 4.10. *Let $U$ be a finitely generated semigroup and $N$ be a subsemigroup of $U$ such that $Q = U \backslash N$ is finite. Then $N$ is a finitely generated semigroup.*

*Proof.* Since $U$ is finitely generated, there exists a finite subset $S$ of $N$ such that $S \cup Q$ is a set of generators of $U$. Let

$$G = S \cup [(Q^2 \cup Q^3 \cup SQ \cup QS \cup QSQ) \cap N].$$

Then, $G$ is a finite subset of $N$.

Let $u \in N$. Then $u = y_1 \cdots y_n$ with $y_i \in S \cup Q$. We prove, by induction on $n$, that $u \in \langle G \rangle$.

If $n = 1$, then $u \in S$ and $u \in \langle G \rangle$. Assume that the conclusion holds for every $k$ such that $1 \leqslant k \leqslant n$. Let $u = y_1 \cdots y_{n+1}$ such that $u \in N$ and $y_i \in S \cup Q$.

*First case.* Assume that $y_{n+1} \in S$.

(a) If $y_1 \cdots y_n \in Q$ then $u \in QS \cap N$. Thus $u \in \langle G \rangle$.

(b) If $y_1 \cdots y_n \in N$ then, by hypothesis of induction, $y_1 \cdots y_n \in \langle G \rangle$. Since $y_{n+1} \in S$ and $S \subset \langle G \rangle$, $u \in \langle G \rangle$.

*Second case.* Assume that $y_{n+1} \in Q$. Let $k$ be the smallest integer such that $y_{k+1} \cdots y_{n+1} \in Q$ (note that $0 \leqslant k \leqslant n$).

(a) If $k = 0$ or if $y_1 \cdots y_{k-1} \in Q$, we have

$$u = \underbrace{y_1 \cdots y_{k-1}}_{1 \cup Q} \underbrace{y_k}_{Q \cup S} \underbrace{y_{k+1} \cdots y_{n+1}}_{Q}.$$

Thus $u \in [(1 \cup Q)(Q \cup S)Q] \cap N$, i.e., $u \in (Q^2 \cup Q^3 \cup SQ \cup QSQ) \cap N$ and $u \in \langle G \rangle$.

(b) In the other case, $1 \leqslant k \leqslant n$ and $y_1 \cdots y_{k-1} \in N$. Since $k$ is the smallest integer such that $y_{k+1} \cdots y_{n+1} \in Q$, $y_k \cdots y_{n+1} \in N$. By hypothesis of induction, $y_1 \cdots y_{k-1} \in \langle G \rangle$ and $y_k \cdots y_{n+1} \in \langle G \rangle$, hence $u \in \langle G \rangle$.

Thus $G$ is a finite set of generators of $N$. ∎

COROLLARY 4.3. *Let $N$ be a semigroup, $F$ a finite semigroup and $U$ an ideal extension of $N$ by $F$. If $U$ is finitely generated then $N$ is finitely generated.*

*Proof of Proposition* 4.3. By induction on $n$. Assume that any finitely generated element of $\mathscr{F}_n$ is a rational thin semigroup. Let $U$ be a finitely generated element of $\mathscr{F}_{n+1}$, then $U$ is an ideal or a Redei extension of an element $M$ of $\mathscr{F}_n$ by a finite semigroup. By Corollary 4.3 or Lemma 4.9, $M$ is finitely generated. By hypothesis of induction, $M$ is thus a rational thin semigroup. Hence, $U$ is a rational thin semigroup (Lemma 4.7 or Lemma 4.8). ∎

The finitely generated elements of the family $\mathscr{F}$ do not exhaust the family of all rational thin semigroups. For instance, the quotient of $\{a, b\}^*$ by the congruence generated by $ab = ba = a$ is a rational thin monoid which is not in $\mathscr{F}$.

It is a consequence of Corollary 4.2 and Proposition 4.3 that an element of $\mathscr{F}$ cannot be the syntactic semigroup of an algebraic language since the syntactic semigroup of a language is finitely generated.

In his paper, Perrot (1977) defined the family of "monoïdes filiformes": a monoid $U$ is "*filiforme*" if it contains an ideal $M$ such that

— $M$ is a Redei extension of $\mathbf{N}_+$ by a finite semigroup $F$

— $Q = U \backslash M$ is finite.

Clearly, the "monoïdes filiformes" are all elements of $\mathscr{F}$ (of $\mathscr{F}_2$, indeed) and the result of Perrot is a consequence of Proposition 4.3 and Corollary 4.2.

RECEIVED October 18, 1988; FINAL MANUSCRIPT RECEIVED June 1, 1989

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